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Couplings of a collection of BF models to matter theories

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Abstract. The couplings of a collection of BF models to matter theories are addressed in the framework of the antifield-BRST deformation procedure. The general theory is exemplified in the case where the matter fields are a set of Dirac spinors and respectively a collection of real scalar fields.

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1 Introduction

The power of the BRST formalism was strongly increased by its cohomological development, which allowed, among others, a useful investigation of many interesting aspects related to the perturbative renormalization problem [1–3], the anomaly-tracking mechanism [3,4], the simultaneous study of local and rigid invariances of a given theory [5], as well as to the reformulation of the construction of consistent interactions in gauge theories [6] in terms of the deformation theory [7,8], or, actually, in terms of the deformation of the solution to the master equation.

The main aim of this paper is to construct all consistent Lagrangian interactions in four spacetime dimensions that can be added to a "free" model that describes a generic matter theory uncoupled to a collection of abelian BF models [9] by means of deforming the solution to the master equation with the help of specific cohomological techniques. The field sector of the four-dimensional BF model consists of a collection of scalar fields, two sets of vector fields and a system of two-forms. Interacting topological field theories of BF-type are important in view of their relationship with Poisson sigma models, which are known to explain interesting aspects of two-dimensional gravity, including the study of classical solutions [10]. Other interesting aspects envisaging BF models can be found in [11] from a Lagrangian perspective and also in [12] from a Hamiltonian point of view. The results presented here extend our former work [13] on the Lagrangian couplings between a sole BF model and matter fields.

The couplings are obtained on the grounds of smoothness, locality, (background) Lorentz invariance and Poincaré invariance. In addition, we require the conservation of the number of derivatives on each field in order to prevent any changes in the differential order of the field equations with respect to the "free" model. The entire Lagrangian formulation of the interacting theory is obtained from the computation of the deformed solution to the master equation, order by order in the coupling constant g. The existence of consistent couplings of order g between the matter fields and the BF ones is ensured under the supplementary, reasonable hypothesis that the matter theory is invariant under some (non-trivial) bosonic global symmetries, which produce some (non-trivially) conserved currents j_a^μ. It is essential that the number of rigid symmetries is equal to the number of BF fields from the collection. Based on the derivative order assumption, we argue that the generators of the rigid symmetries cannot involve the derivatives of the matter fields, and consequently we take them to be linear in these fields, with some coefficients that are the elements of a set of constant matrices T_a. The consistency of the deformation procedure at order g^2 requires the existence of some antisymmetric functions W_{ab} that depend only on the undifferentiated scalar fields, which have the meaning of the components of the two-tensor on a Poisson manifold with the scalar fields viewed as the local coordinates on the target space, and holds in two situations. In the first case (type I solution) all the matrices T_a commute and in the second one (type II solution) their commutators close according to a Lie algebra $L(\mathcal{G})$ with the structure constants $-\bar{f}_{bc}^a$. Type II solutions also restrict the components of the two-tensor W_{ab} to be polynomials of order one in the scalar fields, with the coefficients of the linear terms precisely $\bar{f}_{bc}^a$. The deformation procedure stops at order one if the matter currents j_a^μ include no derivatives and if they either remain invariant under the gauge version of the rigid symmetries in the first case or transform under the gauge version according to the adjoint representation of $L(\mathcal{G})$ in the second case. Otherwise, there appear deformations of order g^2 and possibly of higher orders. It turns out

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that our procedure deforms everything, namely, the Lagrangian action, its gauge transformations and also the accompanying gauge algebra.

This paper is organized into six sections. Section 2 introduces the model to be considered and constructs its "free" Lagrangian BRST symmetry. Section 3 briefly reviews the procedure of adding consistent interactions in gauge theories based on the deformation of the solution to the master equation. In Sect. 4 we construct the Lagrangian interactions for the starting "free" system in four dimensions by solving the deformation equations with the help of standard cohomological techniques. Section 5 applies the theoretical part of the paper to the case where the role of the matter fields is played by a set of Dirac fields and respectively by a collection of real scalar fields, and Sect. 6 ends the paper with the main conclusions.

2 "Free" BRST symmetry

We begin with a "free" theory in four spacetime dimensions, described by the sum between a collection of BF-like Lagrangian actions and a matter action,

$$\begin{aligned} &S_0\left[A^a_\mu, H^a_\mu, \varphi_a, B^{\mu\nu}_a, y^i\right] \\ &= \int \mathrm{d}^4x \left(H^a_\mu \partial^\mu \varphi_a + \frac{1}{2} B^{\mu\nu}_a \partial_{[\mu} A^a_{\nu]} + \mathcal{L}_0\left(\left[y^i\right]\right)\right) \\ &\equiv S^{\mathrm{BF}}_0\left[A^a_\mu, H^a_\mu, \varphi_a, B^{\mu\nu}_a\right] + S^{\mathrm{matt}}_0\left[y^i\right], \end{aligned} \quad (1)$$

where the BF field spectrum contains a set of two-forms $\{B^{\mu\nu}_a\}$, two systems of one-forms $\{A^a_\mu, H^a_\mu\}$, and a collection of scalar fields $\{\varphi_a\}$. The discrete index a is an integer valued from 1 to N, and the number of matter fields is denoted by I ($i = \overline{1, I}$). Here and in the sequel the notation $f([q])$ signifies that f depends on q and its spacetime derivatives up to a finite order, and $[\mu\nu\cdots]$ (or possibly $[ab\cdots]$) means full antisymmetrization with respect to the indices between brackets such that all the independent terms appear only once and are not multiplied by additional numerical factors. We assume that the Lagrangian density $\mathcal{L}_0$ is however no more than second order in the derivatives of the matter fields y^i and that it displays no non-trivial gauge symmetries. In what follows ε_i denotes the Grassmann parity of the matter field y^i. The action (1) is found to be invariant under the gauge transformations (generating set)

$$\delta_\epsilon A^a_\mu = \partial_\mu \epsilon^a \equiv \left(R^{(A)a}_\mu\right)_b \epsilon^b,$$

$$\delta_\epsilon H^a_\mu = 2\partial^\nu \epsilon^a_{\mu\nu} \equiv \left(R^{(H)a}_\mu\right)^{\nu\rho}_b \epsilon^b_{\nu\rho}, \quad (2)$$

$$\delta_\epsilon \varphi_a = 0,\ \delta_\epsilon B^{\mu\nu}_a = -3\partial_\rho \epsilon^{\mu\nu\rho}_a \equiv \left(R^{(B)\mu\nu}_a\right)^b_{\rho\lambda\sigma} \epsilon^{\rho\lambda\sigma}_b,$$

$$\delta_\epsilon y^i = 0, \quad (3)$$

that are abelian and off-shell, second-stage reducible. The gauge parameters are all bosonic, with $\epsilon^a_{\mu\nu}$ and $\epsilon^{\mu\nu\rho}_a$ completely antisymmetric in their Lorentz indices. The redundancy of the gauge generators of the fields H^a_μ and $B^{\mu\nu}_a$ decouples at the level of the "free" model, as it can be seen from the reducibility relations

$$\left(R^{(H)a}_\mu\right)^{\nu\rho}_b \left(Z^{(1)b}_{\nu\rho}\right)^{\lambda\sigma\tau}_c = 0,$$

$$\left(R^{(B)\mu\nu}_a\right)^b_{\rho\lambda\sigma} \left(Z^{(1)\rho\lambda\sigma}_b\right)^c_{\tau\kappa\varsigma\xi} = 0, \quad (4)$$

$$\left(Z^{(1)b}_{\nu\rho}\right)^{\lambda\sigma\tau}_c \left(Z^{(2)c}_{\lambda\sigma\tau}\right)^{\kappa\varsigma\xi\theta}_d = 0, \quad (5)$$

where the reducibility functions respectively take the form

$$\left(Z^{(1)b}_{\nu\rho}\right)^{\lambda\sigma\tau}_c = -\frac{1}{2}\delta^b_c \delta^{[\lambda}_\nu \delta^\sigma_\rho \partial^{\tau]}, \quad (6)$$

$$\left(Z^{(1)\rho\lambda\sigma}_b\right)^c_{\tau\kappa\varsigma\xi} = \frac{1}{6}\delta^c_b \delta^\rho_{[\tau} \delta^\lambda_\kappa \delta^\sigma_\varsigma \partial_{\xi]}, \quad (7)$$

$$\left(Z^{(2)c}_{\lambda\sigma\tau}\right)^{\kappa\varsigma\xi\theta}_d = \frac{1}{6}\delta^c_d \delta^{[\kappa}_\lambda \delta^\varsigma_\sigma \delta^\xi_\tau \partial^{\theta]}. \quad (8)$$

All these properties can be synthesized by the statement that the "free" model under discussion is a so-called "normal" gauge theory, of Cauchy order equal to four. In particular, the matter action $S^{\mathrm{matt}}_0\left[y^i\right]$ is assumed to define a theory of Cauchy order equal to one, while the BF model alone, with the action $S^{\mathrm{BF}}_0\left[A^a_\mu, H^a_\mu, \varphi_a, B^{\mu\nu}_a\right]$, is described by a linear gauge theory of Cauchy order equal to four.

With the purpose of constructing all consistent deformations of this theory in mind, we initially determine its BRST symmetry. The BRST algebra is generated by the field, ghost and antifield spectra

$$\varPhi^{\alpha_0} = \left(A^a_\mu, H^a_\mu, \varphi_a, B^{\mu\nu}_a, y^i\right),$$

$$\varPhi^*_{\alpha_0} = \left(A^{*\mu}_a, H^{*\mu}_a, \varphi^{*a}, B^{*a}_{\mu\nu}, y^*_i\right), \quad (9)$$

$$\eta^{\alpha_1} = \left(\eta^a, C^a_{\mu\nu}, \eta^{\mu\nu\rho}_a\right),\ \eta^*_{\alpha_1} = \left(\eta^*_a, C^{*\mu\nu}_a, \eta^{*a}_{\mu\nu\rho}\right), \quad (10)$$

$$\eta^{\alpha_2} = \left(C^a_{\mu\nu\rho}, \eta^{\mu\nu\rho\lambda}_a\right),\ \eta^*_{\alpha_2} = \left(C^{*\mu\nu\rho}_a, \eta^{*a}_{\mu\nu\rho\lambda}\right), \quad (11)$$

$$\eta^{\alpha_3} = C^a_{\mu\nu\rho\lambda},\ \eta^*_{\alpha_3} = C^{*\mu\nu\rho\lambda}_a, \quad (12)$$

where the fermionic ghosts η^{α_1} and η^{α_3}correspond to the gauge parameters, respectively, to the second-order reducibility relations, while the bosonic ghosts η^{α_2} are due to the first-order redundancy of the generating set. The star variables denote the antifields and exhibit statistics opposite to that of the corresponding fields/ghosts.

Since both the gauge generators and the reducibility functions are field independent, it follows that the BRST differential reduces to

$$s = \delta + \gamma, \quad (13)$$

where δ is the Koszul–Tate differential, and γ means the exterior longitudinal derivative. The Koszul–Tate differential is graded in terms of the antighost number (agh, $\mathrm{agh}(\delta) = -1$) and enforces a resolution of the algebra of smooth functions defined on the stationary surface of field equations for action (1), $C^\infty(\varSigma)$, $\varSigma : \delta S_0/\delta\varPhi^{\alpha_0} = 0$. The exterior longitudinal derivative is graded in terms of the pure ghost number (pgh, $\mathrm{pgh}(\gamma) = 1$) and is correlated with the original gauge symmetry via its cohomology in pure ghost number zero computed in $C^{\infty}(\Sigma)$, which is isomorphic to the algebra of physical observables for this "free" theory. The degrees of the generators (9)–(12) from the BRST complex are valued like

$$\begin{aligned}
&\mathrm{pgh}(\Phi^{\alpha_0}) = \mathrm{pgh}(\Phi^{*}_{\alpha_0}) = \mathrm{pgh}(\eta^{*}_{\alpha_1}) = 0,\\
&\mathrm{pgh}(\eta^{\alpha_1}) = 1, \qquad (14)\\
&\mathrm{agh}(\Phi^{\alpha_0}) = \mathrm{agh}(\eta^{\alpha_1}) = 0,\ \mathrm{agh}(\Phi^{*}_{\alpha_0}) = 1,\\
&\mathrm{agh}(\eta^{*}_{\alpha_1}) = 2, \qquad (15)\\
&\mathrm{pgh}(\eta^{\alpha_2}) = 2,\ \mathrm{pgh}(\eta^{\alpha_3}) = 3,\\
&\mathrm{pgh}(\eta^{*}_{\alpha_2}) = \mathrm{pgh}(\eta^{*}_{\alpha_3}) = 0, \qquad (16)\\
&\mathrm{agh}(\eta^{\alpha_2}) = \mathrm{agh}(\eta^{\alpha_3}) = 0,\ \mathrm{agh}(\eta^{*}_{\alpha_2}) = 3,\\
&\mathrm{agh}(\eta^{*}_{\alpha_3}) = 4, \qquad (17)
\end{aligned}$$

and the actions of δ and γ on them are given by

$$\begin{aligned}
\delta\Phi^{\alpha_0} &= \delta\eta^{\alpha_1} = \delta\eta^{\alpha_2} = \delta\eta^{\alpha_3} = 0, \qquad (18)\\
\delta y^{*}_{i} &= -\frac{\delta^{L}\mathcal{L}_0}{\delta y^{i}},\ \delta A^{*\mu}_{a} = \partial_\nu B^{\nu\mu}_{a},\ \delta H^{*\mu}_{a} = -\partial^{\mu}\varphi_a,\\
\delta\varphi^{*a} &= \partial^{\mu}H^{a}_{\mu}, \qquad (19)\\
\delta B^{*a}_{\mu\nu} &= -\frac{1}{2}\partial_{[\mu}A^{a}_{\nu]},\ \delta\eta^{*}_{a} = -\partial_\mu A^{*\mu}_{a},\\
\delta C^{*\mu\nu}_{a} &= \partial^{[\mu}H^{*\nu]}_{a}, \qquad (20)\\
\delta\eta^{*a}_{\mu\nu\rho} &= \partial_{[\mu}B^{*a}_{\nu\rho]},\ \delta C^{*\mu\nu\rho}_{a} = -\partial^{[\mu}C^{*\nu\rho]}_{a}, \qquad (21)\\
\delta\eta^{*a}_{\mu\nu\rho\lambda} &= -\partial_{[\mu}\eta^{*a}_{\nu\rho\lambda]},\ \delta C^{*\mu\nu\rho\lambda}_{a} = \partial^{[\mu}C^{*\nu\rho\lambda]}_{a}, \qquad (22)\\
\gamma\Phi^{*}_{\alpha_0} &= \gamma\eta^{*}_{\alpha_1} = \gamma\eta^{*}_{\alpha_2} = \gamma\eta^{*}_{\alpha_3} = 0, \qquad (23)\\
\gamma A^{a}_{\mu} &= \partial_\mu\eta^{a},\ \gamma H^{a}_{\mu} = 2\partial^{\nu}C^{a}_{\mu\nu},\ \gamma\varphi_a = 0,\\
\gamma B^{\mu\nu}_{a} &= -3\partial_\rho\eta^{\mu\nu\rho}_{a}, \qquad (24)\\
\gamma y^{i} &= 0,\ \gamma\eta^{a} = 0,\ \gamma C^{a}_{\mu\nu} = -3\partial^{\rho}C^{a}_{\mu\nu\rho},\\
\gamma\eta^{\mu\nu\rho}_{a} &= 4\partial_\lambda\eta^{\mu\nu\rho\lambda}_{a}, \qquad (25)\\
\gamma C^{a}_{\mu\nu\rho} &= 4\partial^{\lambda}C^{a}_{\mu\nu\rho\lambda},\ \gamma\eta^{\mu\nu\rho\lambda}_{a} = 0,\ \gamma C^{a}_{\mu\nu\rho\lambda} = 0. \qquad (26)
\end{aligned}$$

The overall degree from the BRST complex is named the ghost number (gh) and is defined like the difference between the pure ghost number and the antighost number, such that $\mathrm{gh}(\delta) = \mathrm{gh}(\gamma) = \mathrm{gh}(s) = 1$. The BRST symmetry admits a canonical action $s\cdot = (\cdot, S)$, where its canonical generator ($\mathrm{gh}(S) = 0$, $\varepsilon(S) = 0$) satisfies the classical master equation

$$(S,S) = 0. \qquad (27)$$

The antibracket $(,)$ is obtained by decreeing the fields/ghosts respectively conjugated with the corresponding antifields. In the case of the "free" theory under discussion, the solution to the master equation takes the form

$$S = S_0 + \int \mathrm{d}^4x \left(A^{*\mu}_{a}\partial_\mu\eta^{a} + 2H^{*\mu}_{a}\partial^{\nu}C^{a}_{\mu\nu} - 3B^{*a}_{\mu\nu}\partial_\rho\eta^{\mu\nu\rho}_{a} - 3C^{*\mu\nu}_{a}\partial^{\rho}C^{a}_{\mu\nu\rho} + 4\eta^{*a}_{\mu\nu\rho}\partial_\lambda\eta^{\mu\nu\rho\lambda}_{a} + 4C^{*\mu\nu\rho}_{a}\partial^{\lambda}C^{a}_{\mu\nu\rho\lambda}\right), \qquad (28)$$

and it contains pieces of antighost number ranging from zero to three.

3 Basic equations of the deformation procedure

We consider the problem of constructing the consistent interactions that can be added to the "free" Lagrangian action (1), $S_0[\Phi^{\alpha_0}]$ (where Φ^{α_0} means the original field spectrum in (9)), invariant under the gauge transformations (2) and (3), written in a compact form like

$$\delta_\epsilon\Phi^{\alpha_0} = R^{\alpha_0}_{\ \ \alpha_1}\epsilon^{\alpha_1},\ \frac{\delta S_0}{\delta\Phi^{\alpha_0}}R^{\alpha_0}_{\ \ \alpha_1} = 0, \qquad (29)$$

such that the couplings preserve both the field spectrum and the original number of independent gauge symmetries. This issue is addressed by means of reformulating the problem of constructing consistent interactions like a deformation problem of the solution (28) to the master equation corresponding to the "free" theory [7]. Such a reformulation is possible due to the fact that the solution to the master equation contains all the information on the gauge structure of the theory. If a consistent interacting gauge theory can be constructed, then the solution S to the master equation associated with the "free" theory, $(S,S) = 0$, can be deformed into a solution $\bar{S}$,

$$\begin{aligned}
S \to \bar{S} &= S + gS_1 + g^2S_2 + \cdots\\
&= S + g\int \mathrm{d}^4x\, a + g^2\int \mathrm{d}^4x\, b + \cdots, \qquad (30)
\end{aligned}$$

of the master equation for the deformed theory

$$(\bar{S},\bar{S}) = 0, \qquad (31)$$

such that both the field/ghost and antifield spectra of the initial theory, (9)–(12), are preserved. Equation (31) splits, according to the various orders in the coupling constant (deformation parameter) g, into (27) and

$$\begin{aligned}
2(S_1,S) &= 0, \qquad (32)\\
2(S_2,S) + (S_1,S_1) &= 0, \qquad (33)\\
(S_3,S) + (S_1,S_2) &= 0, \qquad (34)\\
&\vdots
\end{aligned}$$

Equation (27) is fulfilled by hypothesis. The next one, (32), requires that the first-order deformation of the solution to the master equation, S_1, is a cocycle of the "free" BRST differential s. However, only cohomologically nontrivial solutions to (32) should be taken into account, since the BRST-exact ones can be eliminated by a (in general non-linear) field redefinition. This means that S_1 pertains to the ghost number zero cohomological space of s, $H^0(s)$, which is generically non-empty due to its isomorphism to the space of physical observables of the "free" theory. It has been shown in [7] due to the triviality of the antibracket map in the cohomology of the BRST differential that there are no obstructions in finding solutions to the remaining equations ((33), (34) etc.). Unfortunately, the resulting interactions may be non-local, and there might even appear obstructions if one insists on their locality.

The analysis of these obstructions can be done with the help of cohomological techniques. As it will be seen below, all the consistent interactions in the case of the model under study turn out to be local.

4 Deformation of the "free" solution to the master equation

Here, we compute the consistent Lagrangian interactions that can be added to the "free" model analyzed so far, which describes a generic matter theory plus a topological BF-type model in four spacetime dimensions. This is achieved by solving the deformation equations (32)–(34) etc., with the help of some specific cohomological techniques. Our aim is to determine the complete deformed solution to the master equation, which is consistent to all orders in the coupling constant. For obvious reasons, we consider only smooth, local, (background) Lorentz invariant and, moreover, Poincaré invariant quantities (i.e. we do not allow explicit dependence on the spacetime coordinates). The smoothness of the deformations refers to the fact that the deformed solution (30) to the master equation is smooth in the coupling constant g and reduces to the original solution (28) in the free limit $g = 0$. In addition, we require the conservation of the number of derivatives on each field (this condition is frequently met in the literature; for instance, see [14–16]). The last requirement will be brought in only after the derivation of the general form of the first-order deformation, during the consistency procedure.

4.1 First-order deformation

4.1.1 Standard material: $H(\gamma)$ and $H(\delta|d)$

Initially, we approach the first-order deformation of the solution to the master equation, described by (32). Using the notation from (30), its local form is

$$sa = \partial_\mu m^\mu, \tag{35}$$

for some local current m^μ, so it requires that a is a s-cocycle modulo the exterior spacetime differential d. In order to analyze the above equation, we develop a according to the antighost number

$$a = \sum_{k=0}^{J} a_k,\ \mathrm{agh}(a_k) = k,\ \mathrm{gh}(a_k) = 0,\ \varepsilon(a_k) = 0, \tag{36}$$

and assume, without loss of generality, that a stops at some finite value J of the antighost number. This can be shown, for instance, like in [14] (Sect. 3), under the sole assumption that the interacting Lagrangian at the first order in the coupling constant, a_0, has a finite, but otherwise arbitrary derivative order. By taking into account the decomposition (13) of the BRST differential, (35) is equivalent to a tower of local equations, corresponding to the various decreasing values of the antighost number

$$\gamma a_J = \partial_\mu \overset{(J)^{\mu}}{m}, \tag{37}$$

$$\delta a_J + \gamma a_{J-1} = \partial_\mu \overset{(J-1)^{\mu}}{m}, \tag{38}$$

$$\delta a_k + \gamma a_{k-1} = \partial_\mu \overset{(k-1)^{\mu}}{m},\ 1 \leq k \leq J-1, \tag{39}$$

where $\left(\overset{(k)^{\mu}}{m}\right)_{k=\overline{0,J}}$ are some local currents, with $\mathrm{agh}\left(\overset{(k)^{\mu}}{m}\right) = k$. It can be proved that (37) can be replaced at strictly positive values of the antighost number by

$$\gamma a_J = 0,\ J > 0. \tag{40}$$

The proof of this result is standard material and can be found for instance in [14–17]. Then, in order to solve (35) (equivalent with (40), and (38) and (39)) we clearly need to compute the cohomology of γ, $H(\gamma)$. Due to (23)–(26) it is simple to see that $H(\gamma)$ is spanned by

$$\omega^{\Delta} = \left(F^a_{\mu\nu} = \partial_{[\mu} A^a_{\nu]}, \partial^\mu H^a_\mu, \varphi_a, \partial_\nu B^{\mu\nu}_a, y^i\right), \tag{41}$$

$$\chi^* = \left(\Phi^*_{\alpha_0}, \eta^*_{\alpha_1}, \eta^*_{\alpha_2}, \eta^*_{\alpha_3}\right), \tag{42}$$

and by their spacetime derivatives, as well as by the undifferentiated ghosts

$$\eta^{A_1} = \left(\eta^a, \eta^{\mu\nu\rho\lambda}_a, C^a_{\mu\nu\rho\lambda}\right). \tag{43}$$

(The derivatives of the ghosts η^{A_1} are removed from $H(\gamma)$ since they are γ-exact, in agreement with the first relation in (24), the last formula in (25) and respectively the first definition from (26).) If we denote by $e^M\left(\eta^{A_1}\right)$ the elements with the pure ghost number equal to M of a basis in the space of polynomials in the ghosts η^{A_1}, it follows that the general, local solution to (40) takes the form (up to trivial γ-exact contributions)

$$a_J = \mu_J\left(\left[\omega^{\Delta}\right], \left[\chi^*\right]\right) e^J\left(\eta^{A_1}\right), \tag{44}$$

where $\mathrm{agh}(\mu_J) = J$ and $\mathrm{pgh}\left(e^J\right) = J$. The objects μ_J (obviously non-trivial in $H^0(\gamma)$) were taken to have a bounded number of derivatives and a finite antighost number, so they are polynomials in the antifields χ^*, in their derivatives, in all the quantities ω^{Δ} excepting the undifferentiated scalar fields φ_a and the undifferentiated bosonic matter fields y^i (if any), as well as in all of their derivatives. However, the μ_J's may in principle have coefficients that are infinite series in φ_a and in all commuting y^i. Due to their γ-closeness and (partial) polynomial character, μ_J will be called "invariant polynomials". A useful property is that the cohomology of d in the space of invariant polynomials is trivial in form degree strictly less than four and in strictly positive antighost number (for a general proof, see [18]). This further leads to the conclusion that there is no non-trivial descent for $H(\gamma|d)$ in strictly positive antighost number, or, to put it otherwise, that (37) can always be replaced with (40) for $J>0$. The proof to the last result can be done as in [15–17].

Replacing (44) in (38) we remark that a necessary (but not sufficient) condition for the existence of (non-trivial) solutions a_{J-1} is that the invariant polynomials μ_J from (44) are (non-trivial) objects from the local cohomology of the Koszul–Tate differential $H(\delta|d)$ in antighost number $J>0$ and in pure ghost number equal to zero, $\mu_J \in H_J(\delta|d)$, i.e.

$$\delta\mu_J = \partial_\mu \overset{(J-1)^\mu}{j}, \ \text{agh}\left(\overset{(J-1)^\mu}{j}\right) = J-1,$$
$$\text{pgh}\left(\overset{(J-1)^\mu}{j}\right) = 0. \tag{45}$$

Consequently, we need to investigate some of the main properties of the local cohomology of the Koszul–Tate differential in pure ghost number zero and in strictly positive antighost numbers[1] in order to fully determine the component a_J of highest antighost number from the first-order deformation. To this end we observe that the form (1) of the "free" Lagrangian action together with the definitions (18)–(22) enable us to analyze $H_k(\delta|d)$ in terms of the local cohomologies $H_k^{\text{matt}}(\delta|d)$ and $H_k^{\text{BF}}(\delta|d)$, where the last local cohomologies in antighost number k refer to the Koszul–Tate operator that acts non-trivially only in the matter sector, respectively, only in the BF one[2]. In the light of the general results from [19], according to which the local cohomology of the Koszul–Tate differential in pure ghost number zero for a given gauge theory is trivial in antighost numbers strictly greater than the Cauchy order of this theory, combined with the fact that the separate Cauchy orders of the matter theory and of the BF model are equal to one and respectively to four, we can state that

$$H_k^{\text{matt}}(\delta|d) = 0,\ k>1, \tag{46}$$
$$H_k^{\text{BF}}(\delta|d) = 0,\ k>4. \tag{47}$$

By means of the above results it follows that

$$H_k(\delta|d) = 0,\ k>4 \tag{48}$$

for the overall "free" theory (1) and, moreover, that $H_k(\delta|d) = H_k^{\text{BF}}(\delta|d)$ for $k=2,3,4$. As for $H_1(\delta|d)$, this is the only case where the general representative of the local cohomology of the Koszul–Tate differential involves, and possibly mixes, the field/ghost and antifield spectra of both BF and matter sectors. It is quite reasonable to assume that if the invariant polynomial μ_k, with $\text{agh}(\mu_k) = k \geq 4$, is trivial in $H_k(\delta|d)$; then it can be taken to be trivial also in $H_k^{\text{inv}}(\delta|d)$:

$$\left(\mu_k = \delta b_{k+1} + \partial_\mu \overset{(k)^\mu}{c},\ \text{agh}(\mu_k) = k \geq 4\right) \Rightarrow$$
$$\mu_k = \delta\beta_{k+1} + \partial_\mu \overset{(k)^\mu}{\gamma}, \tag{49}$$

where β_{k+1} and $\overset{(k)^\mu}{\gamma}$ are invariant polynomials. [An element of $H_k^{\text{inv}}(\delta|d)$ is defined via an equation similar to (45) for $J \to k$, but with the corresponding current an invariant polynomial.] This assumption is based on what happens in many gauge theories. For instance, see [14–17]. The results (48) and (49) yield the conclusion that all the local cohomology of the Koszul–Tate differential in the space of invariant polynomials in antighost numbers strictly greater than four is indeed trivial:

$$H_k^{\text{inv}}(\delta|d) = 0,\ k>4. \tag{50}$$

The previous results on $H(\delta|d)$ and $H^{\text{inv}}(\delta|d)$ in strictly positive antighost numbers are important because they control the obstructions to removing the antifields from the first-order deformation. This statement is also standard material and can be shown as in [14–17]. Its proof is mainly based on the formulas (48)–(50) and relies on the fact that we can successively eliminate all the pieces of antighost number strictly greater than four from the non-integrated density of the first-order deformation by adding *only* trivial terms. As a consequence, we can safely take the first-order deformation to stop at antighost number four ($J=4$ in the expansion (36)):

$$a = a_0 + a_1 + a_2 + a_3 + a_4, \tag{51}$$

where a_4 is of the form (44) and μ_4 is a non-trivial element from $H_4^{\text{inv}}(\delta|d) = H_4^{\text{invBF}}(\delta|d)$.

4.1.2 Computation of the first-order deformation

After some computation, we infer that the most general, non-trivial representative of $H_4^{\text{invBF}}(\delta|d)$ can be taken to be of the type

$$(\mu_4)^{\mu\nu\rho\lambda} = \left(\frac{\partial U}{\partial\varphi_e} C_e^{*\mu\nu\rho\lambda} + \frac{\partial^2 U}{\partial\varphi_e\partial\varphi_f}\left(H_e^{*[\mu} C_f^{*\nu\rho\lambda]} + C_e^{*[\mu\nu} C_f^{*\rho\lambda]}\right) + \frac{\partial^3 U}{\partial\varphi_e\partial\varphi_f\partial\varphi_g} H_e^{*[\mu} H_f^{*\nu} C_g^{*\rho\lambda]} + \frac{\partial^4 U}{\partial\varphi_e\partial\varphi_f\partial\varphi_g\partial\varphi_h} H_e^{*\mu} H_f^{*\nu} H_g^{*\rho} H_h^{*\lambda}\right), \tag{52}$$

where $U = U(\varphi_a)$ is an arbitrary function depending only on the *undifferentiated* scalar fields. We omit the proof of this result, which is rather tedious and not illuminating,

---

[1] We recall that the local cohomology $H(\delta|d)$ is completely trivial in both strictly positive antighost *and* pure ghost numbers (for instance, see [19], Theorem 5.4 and [20]).

[2] Indeed, we can decompose δ like $\delta = \delta^{\text{matt}} + \delta^{\text{BF}}$, where $\delta^{\text{matt}}(\text{matter variables}) = \delta(\text{matter variables})$ and $\delta^{\text{matt}}(\text{BF variables}) = 0$, respectively, $\delta^{\text{BF}}(\text{matter variables}) = 0$ and $\delta^{\text{BF}}(\text{BF variables}) = \delta(\text{BF variables})$. According to this decomposition, $H^{\text{matt}}(\delta|d)$ and $H^{\text{BF}}(\delta|d)$ must be understood only like some more suggestive notation for $H(\delta^{\text{matt}}|d)$ and $H(\delta^{\text{BF}}|d)$ respectively.

but observe that it is in perfect agreement with the similar one given in [13] in the absence of BF collection indices and also with that resulting from the Hamiltonian analysis of the action (1) in n dimensions and reported in [12]. On the other hand, the elements of pure ghost number equal to four of a basis in the space of the polynomials in the ghosts (43) are

$$\eta^a C^b_{\mu\nu\rho\lambda},\ \eta^a\eta^b\eta_{c\mu\nu\rho\lambda},\ \eta^a\eta^b\eta^c\eta^d,\ \eta_{a\mu\nu\rho\lambda}\eta_{b\mu'\nu'\rho'\lambda'}. \tag{53}$$

Thus, the last representative from the expansion (51) is provided by directly "gluing" (52) to the elements from (53) by means of some appropriate functions of the scalar fields, and hence will be expressed by

$$\begin{aligned}
a_4 = &\left(\frac{\partial W_{ab}}{\partial\varphi_c}C^{*\mu\nu\rho\lambda}_c\right.\\
&+\frac{\partial^2 W_{ab}}{\partial\varphi_c\partial\varphi_d}\left(H^{*[\mu}_c C^{*\nu\rho\lambda]}_d + C^{*[\mu\nu}_c C^{*\rho\lambda]}_d\right)\\
&+\frac{\partial^3 W_{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e}H^{*[\mu}_c H^{*\nu}_d C^{*\rho\lambda]}_e\\
&+\left.\frac{\partial^4 W_{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e\partial\varphi_f}H^{*\mu}_c H^{*\nu}_d H^{*\rho}_e H^{*\lambda}_f\right)\eta^a C^b_{\mu\nu\rho\lambda}\\
&-\frac{1}{4}\left(\frac{\partial M^c_{ab}}{\partial\varphi_d}C^{*\mu\nu\rho\lambda}_d\right.\\
&+\frac{\partial^2 M^c_{ab}}{\partial\varphi_d\partial\varphi_e}\left(H^{*[\mu}_d C^{*\nu\rho\lambda]}_e + C^{*[\mu\nu}_d C^{*\rho\lambda]}_e\right)\\
&+\frac{\partial^3 M^c_{ab}}{\partial\varphi_d\partial\varphi_e\partial\varphi_f}H^{*[\mu}_d H^{*\nu}_e C^{*\rho\lambda]}_f\\
&+\left.\frac{\partial^4 M^c_{ab}}{\partial\varphi_d\partial\varphi_e\partial\varphi_f\partial\varphi_g}H^{*\mu}_d H^{*\nu}_e H^{*\rho}_f H^{*\lambda}_g\right)\eta^a\eta^b\eta_{c\mu\nu\rho\lambda}\\
&+\frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\left(\left(\frac{\partial M^{ab}}{\partial\varphi_c}C^{*\mu\nu\rho\lambda}_c\right.\right.\\
&+\frac{\partial^2 M^{ab}}{\partial\varphi_c\partial\varphi_d}\left(H^{*[\mu}_c C^{*\nu\rho\lambda]}_d + C^{*[\mu\nu}_c C^{*\rho\lambda]}_d\right)\\
&+\frac{\partial^3 M^{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e}H^{*[\mu}_c H^{*\nu}_d C^{*\rho\lambda]}_e\\
&+\left.\frac{\partial^4 M^{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e\partial\varphi_f}H^{*\mu}_c H^{*\nu}_d H^{*\rho}_e H^{*\lambda}_f\right)\eta^{\sigma\tau\kappa\varsigma}_a\eta_{b\sigma\tau\kappa\varsigma}\\
&-\frac{1}{2\cdot(4!)^2}\left(\frac{\partial M_{mnpq}}{\partial\varphi_c}C^{*\mu\nu\rho\lambda}_c\right.\\
&+\frac{\partial^2 M_{mnpq}}{\partial\varphi_c\partial\varphi_d}\left(H^{*[\mu}_c C^{*\nu\rho\lambda]}_d + C^{*[\mu\nu}_c C^{*\rho\lambda]}_d\right)\\
&+\frac{\partial^3 M_{mnpq}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e}H^{*[\mu}_c H^{*\nu}_d C^{*\rho\lambda]}_e\\
&+\left.\left.\frac{\partial^4 M_{mnpq}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e\partial\varphi_f}H^{*\mu}_c H^{*\nu}_d H^{*\rho}_e H^{*\lambda}_f\right)\eta^m\eta^n\eta^p\eta^q\right).
\end{aligned}\tag{54}$$

In the above the functions W_{ab}, M^c_{ab}, M^{ab} and M_{mnpq} depend only on the undifferentiated scalar fields. Meanwhile, M^c_{ab} together with M_{mnpq} are antisymmetric in their lower indices due to the anticommutation among the ghosts η^a, while M^{ab} are symmetric as a consequence of the commutation among the ghosts for ghosts $\eta^{\mu\nu\rho\lambda}_a$. The factors $-1/4$, $+1/2$ and respectively $-1/2\cdot(4!)^2$ in front of the last three pieces were added for further convenience.

By computing the action of δ on a_4 and by taking into account the relations (23)–(26), it follows that the solution a_3 of (38) for $J=4$ is precisely given by

$$\begin{aligned}
a_3 = &\left(\frac{\partial W_{ab}}{\partial\varphi_c}C^{*\mu\nu\rho}_c + \frac{\partial^2 W_{ab}}{\partial\varphi_c\partial\varphi_d}H^{*[\mu}_c C^{*\nu\rho]}_d\right.\\
&+\left.\frac{\partial^3 W_{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e}H^{*\mu}_c H^{*\nu}_d H^{*\rho}_e\right)\left(-\eta^a C^b_{\mu\nu\rho} + 4A^{a\lambda}C^b_{\mu\nu\rho\lambda}\right)\\
&+2\left(6\left(\frac{\partial W_{ab}}{\partial\varphi_c}C^{*\mu\nu}_c + \frac{\partial^2 W_{ab}}{\partial\varphi_c\partial\varphi_d}H^{*\mu}_c H^{*\nu}_d\right)B^{*a\rho\lambda}\right.\\
&+\left.4\frac{\partial W_{ab}}{\partial\varphi_c}H^{*\mu}_c\eta^{*a\nu\rho\lambda} + W_{ab}\eta^{*a\mu\nu\rho\lambda}\right)C^b_{\mu\nu\rho\lambda}\\
&+\frac{1}{2}\left(\frac{\partial M^c_{ab}}{\partial\varphi_d}C^{*\mu\nu\rho}_d + \frac{\partial^2 M^c_{ab}}{\partial\varphi_d\partial\varphi_e}H^{*[\mu}_d C^{*\nu\rho]}_e\right.\\
&+\left.\frac{\partial^3 M^c_{ab}}{\partial\varphi_d\partial\varphi_e\partial\varphi_f}H^{*\mu}_d H^{*\nu}_e H^{*\rho}_f\right)\\
&\times\left(\frac{1}{2}\eta^a\eta^b\eta_{c\mu\nu\rho} - 4A^{a\lambda}\eta^b\eta_{c\mu\nu\rho\lambda}\right)\\
&-\left(6\left(\frac{\partial M^c_{ab}}{\partial\varphi_d}C^{*\mu\nu}_d + \frac{\partial^2 M^c_{ab}}{\partial\varphi_d\partial\varphi_e}H^{*\mu}_d H^{*\nu}_e\right)B^{*a\rho\lambda}\right.\\
&+\left.4\frac{\partial M^c_{ab}}{\partial\varphi_d}H^{*\mu}_d\eta^{*a\nu\rho\lambda} + M^c_{ab}\eta^{*a\mu\nu\rho\lambda}\right)\eta^b\eta_{c\mu\nu\rho\lambda}\\
&-\varepsilon_{\mu\nu\rho\lambda}\left(\frac{\partial M^{ab}}{\partial\varphi_c}C^{*\sigma\tau\kappa}_c + \frac{\partial^2 M^{ab}}{\partial\varphi_c\partial\varphi_d}H^{*[\sigma}_c C^{*\tau\kappa]}_d\right.\\
&+\left.\frac{\partial^3 M^{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e}H^{*\sigma}_c H^{*\tau}_d H^{*\kappa}_e\right)\eta_{a\sigma\tau\kappa}\eta^{\mu\nu\rho\lambda}_b\\
&-\frac{\varepsilon_{\mu\nu\rho\lambda}}{(4!)^2}\left(4\left(\frac{\partial M_{mnpq}}{\partial\varphi_c}C^{*\mu\nu\rho}_c + \frac{\partial^2 M_{mnpq}}{\partial\varphi_c\partial\varphi_d}H^{*[\mu}_c C^{*\nu\rho]}_d\right.\right.\\
&+\left.\frac{\partial^3 M_{mnpq}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e}H^{*\mu}_c H^{*\nu}_d H^{*\rho}_e\right)A^{m\lambda}\\
&+12\left(\frac{\partial M_{mnpq}}{\partial\varphi_c}C^{*\mu\nu}_c + \frac{\partial^2 M_{mnpq}}{\partial\varphi_c\partial\varphi_d}H^{*\mu}_c H^{*\nu}_d\right)B^{*m\rho\lambda}\\
&+\left.8\frac{\partial M_{mnpq}}{\partial\varphi_c}H^{*\mu}_c\eta^{*m\nu\rho\lambda} + 2M_{mnpq}\eta^{*m\mu\nu\rho\lambda}\right)\eta^n\eta^p\eta^q.
\end{aligned}\tag{55}$$

By means of (39) for $k=3$

$$\delta a_3 + \gamma a_2 = \partial_\mu \overset{(2)}{m}{}^\mu, \tag{56}$$

the solution (55) and the definitions (23)–(26) lead to

$$\begin{aligned}
a_2 = &\left(\frac{\partial W_{ab}}{\partial\varphi_c}C^{*\mu\nu}_c + \frac{\partial^2 W_{ab}}{\partial\varphi_c\partial\varphi_d}H^{*\mu}_c H^{*\nu}_d\right)\\
&\times\left(\eta^a C^b_{\mu\nu} - 3A^{a\rho}C^b_{\mu\nu\rho}\right)\\
&-2\left(3\frac{\partial W_{ab}}{\partial\varphi_c}H^{*\mu}_c B^{*a\nu\rho} + W_{ab}\eta^{*a\mu\nu\rho}\right)C^b_{\mu\nu\rho}\\
&-\frac{1}{2}\left(\frac{\partial M^c_{ab}}{\partial\varphi_d}C^{*\mu\nu}_d + \frac{\partial^2 M^c_{ab}}{\partial\varphi_d\partial\varphi_e}H^{*\mu}_d H^{*\nu}_e\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\frac{1}{2}\eta^a\eta^b B_{c\mu\nu} - 3A^{a\rho}\eta^b\eta_{c\mu\nu\rho}\right) \\
&+ \left(3\frac{\partial M^c_{ab}}{\partial\varphi_d}H^{*\mu}_d B^{*a\nu\rho} + M^c_{ab}\eta^{*a\mu\nu\rho}\right)\eta^b\eta_{c\mu\nu\rho} \\
&+ \frac{1}{2}\left(-\frac{\partial M^c_{ab}}{\partial\varphi_d}H^{*\mu}_d A^*_{c\mu} + M^c_{ab}\eta^*_c\right)\eta^a\eta^b \\
&+ \left(3\left(\frac{\partial M^c_{ab}}{\partial\varphi_d}C^{*\mu\nu}_d + \frac{\partial^2 M^c_{ab}}{\partial\varphi_d\partial\varphi_e}H^{*\mu}_d H^{*\nu}_e\right)A^{a\rho}\right. \\
&\left. + 12\frac{\partial M^c_{ab}}{\partial\varphi_d}H^{*\mu}_d B^{*a\nu\rho} + 4M^c_{ab}\eta^{*a\mu\nu\rho}\right)A^{b\lambda}\eta_{c\mu\nu\rho\lambda} \\
&- 6M^c_{ab}B^{*a}_{\mu\nu}B^{*b}_{\rho\lambda}\eta^{\mu\nu\rho\lambda}_c \\
&+ \frac{9}{2}\varepsilon^{\mu\nu\rho\lambda}\left(\frac{\partial M^{ab}}{\partial\varphi_d}C^*_{d\mu\nu} + \frac{\partial^2 M^{ab}}{\partial\varphi_d\partial\varphi_e}H^*_{d\mu}H^*_{e\nu}\right)\eta_{a\rho\sigma\tau}\eta_{b\lambda}{}^{\sigma\tau} \\
&+ \varepsilon_{\mu\nu\rho\lambda}\left(\left(\frac{\partial M^{ab}}{\partial\varphi_d}C^{*\sigma\tau}_d + \frac{\partial^2 M^{ab}}{\partial\varphi_d\partial\varphi_e}H^{*\sigma}_d H^{*\tau}_e\right)B_{b\sigma\tau}\right. \\
&\left. + 2\frac{\partial M^{ab}}{\partial\varphi_d}H^{*\sigma}_d A^*_{b\sigma} - 2M^{ab}\eta^*_b\right)\eta^{\mu\nu\rho\lambda}_a \\
&+ \frac{3\varepsilon_{\mu\nu\rho\lambda}}{(4!)^2}\left(6\left(\frac{\partial M_{mnpq}}{\partial\varphi_d}C^{*\mu\nu}_d + \frac{\partial^2 M_{mnpq}}{\partial\varphi_d\partial\varphi_e}H^{*\mu}_d H^{*\nu}_e\right)\right. \\
&\times A^{m\rho}A^{n\lambda} \\
&+ 24\frac{\partial M_{mnpq}}{\partial\varphi_d}H^{*\mu}_d B^{*m\nu\rho}A^{n\lambda} + 8M_{mnpq}\eta^{*m\mu\nu\rho}A^{n\lambda} \\
&\left. - 12M_{mnpq}B^{*m\mu\nu}B^{*n\rho\lambda}\right)\eta^p\eta^q. \qquad (57)
\end{aligned}
$$

Next, we investigate (39) for $k = 2$:

$$\delta a_2 + \gamma a_1 = \partial_\mu \overset{(1)}{m}{}^\mu, \qquad (58)$$

which, combined with (57), provide a_1:

$$
\begin{aligned}
a_1 = &\frac{\partial W_{ab}}{\partial\varphi_c}H^{*\mu}_c\left(-\eta^a H^b_\mu + 2A^{a\nu}C^b_{\mu\nu}\right) \\
&+ W_{ab}\left(2B^{*a}_{\mu\nu}C^{b\mu\nu} - \varphi^{*a}\eta^b\right) \\
&- \frac{\partial M^c_{ab}}{\partial\varphi_d}H^{*\mu}_d A^{a\nu}\left(\eta^b B_{c\mu\nu} + \frac{3}{2}A^{b\rho}\eta_{c\mu\nu\rho}\right) \\
&- M^c_{ab}\left(B^{*a}_{\mu\nu}\eta^b B^{\mu\nu}_c + A^a_\mu\eta^b A^{*\mu}_c + 3B^{*a}_{\mu\nu}A^b_\rho\eta^{\mu\nu\rho}_c\right) \\
&+ 2\varepsilon_{\nu\rho\sigma\lambda}\left(\frac{\partial M^{ab}}{\partial\varphi_c}H^*_{c\mu}B^{\mu\nu}_a - M^{ab}A^{*\nu}_a\right)\eta^{\rho\sigma\lambda}_b \qquad (59) \\
&+ \frac{\varepsilon^{\mu\nu\rho\lambda}}{4!}\left(\frac{\partial M_{mnpq}}{\partial\varphi_c}H^*_{c\mu}A^m_\nu + 3M_{mnpq}B^{*m}_{\mu\nu}\right)A^n_\rho A^p_\lambda\eta^q + \bar{a}_1,
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{a}_1 = &\left(B^{*a}_{\mu\nu}T^{\mu\nu}_{ab}\left(\left[\omega^\Delta\right]\right) + A^{*a}_\mu\tilde{T}^\mu_{ab}\left(\left[\omega^\Delta\right]\right) + \varphi^{*a}T_{ab}\left(\left[\omega^\Delta\right]\right)\right. \\
&\left. + H^{*\mu}_a T^a_{\mu b}\left(\left[\omega^\Delta\right]\right) + y^*_i\bar{T}^i_b\left(\left[\omega^\Delta\right]\right)\right)\eta^b \\
&\equiv \lambda_b\left(\left[\omega^\Delta\right]\right)\eta^b, \qquad (60)
\end{aligned}
$$

and ω^Δ is explained in (41). In order to produce a bosonic $\bar{a}_1$, as required by the standard rules of the BRST formalism, the gauge invariant functions $T^{\mu\nu}_{ab}$, $\tilde{T}^\mu_{ab}$, T_{ab} and $T^a_{\mu b}$ must be bosonic, while the Grassmann parity of $\bar{T}^i_b$ should be equal to ε_i for each $b = \overline{1,N}$. The term (60) added in the right-hand side of (59) appears to be like the general solution to the "homogeneous" equation $\gamma\bar{a}_1 = 0$ and takes into account the fact that both the BF and the matter theories participate in the local cohomology of the Koszul–Tate differential in antighost number one. Its form is given by the general solution (44) for $J = 1$. Such terms correspond to $\bar{a}_2 = 0$ and thus they do not modify either the gauge algebra or the reducibility functions, but only the gauge transformations of the interacting theory. We emphasize that the solutions a_3 and a_2 obtained previously also include the general ones, corresponding to the "homogeneous" equations $\gamma\bar{a}_3 = 0$ and $\gamma\bar{a}_2 = 0$. In order to simplify the exposition we avoided the discussion regarding the selection procedure of these solutions such as to comply with obtaining some consistent a_2 and a_1. It is however interesting to note that this procedure allows no new functions of the scalar fields beside W_{ab}, M^c_{ab}, M^{ab} and M_{abcd} to enter a_3 or a_2.

In order to solve (39) at antighost number zero,

$$\delta a_1 + \gamma a_0 = \partial^\mu \overset{(0)}{m}_\mu, \qquad (61)$$

whose solution is nothing but the deformed Lagrangian at order one in g, from (59) we observe that

$$
\begin{aligned}
\delta a_1 = &-\gamma\left[-W_{ab}A^{a\mu}H^b_\mu + \frac{1}{2}M^c_{ab}A^a_\mu A^b_\nu B^{\mu\nu}_c\right. \\
&\left. + \frac{1}{2}\varepsilon^{\mu\nu\rho\lambda}\left(M^{ab}B_{a\mu\nu}B_{b\rho\lambda} - \frac{1}{2\cdot 4!}M_{abcd}A^a_\mu A^b_\nu A^c_\rho A^d_\lambda\right)\right] \\
&+ \partial^\mu\left(W_{ab}\left(-\eta^a H^b_\mu + 2A^{a\nu}C^b_{\mu\nu}\right)\right. \\
&- M^c_{ab}A^{a\nu}\left(\eta^b B_{c\mu\nu} + \frac{3}{2}A^{b\rho}\eta_{c\mu\nu\rho}\right) \\
&\left. + 2\varepsilon^{\nu\rho\sigma\lambda}M^{ab}B_{a\mu\nu}\eta_{b\rho\sigma\lambda} + \frac{1}{4!}\varepsilon_{\mu\nu\rho\lambda}M_{abcd}A^{a\nu}A^{b\rho}A^{c\lambda}\eta^d\right) \\
&+ \delta\bar{a}_1. \qquad (62)
\end{aligned}
$$

Thus, the consistency of the deformation procedure at order one in the coupling constant requires that $\delta\bar{a}_1$ must independently be γ-exact modulo d

$$\delta\bar{a}_1 + \gamma\bar{a}_0 = \partial_\mu\bar{j}^\mu. \qquad (63)$$

At the first sight it seems that $\bar{a}_1$ is not an essential ingredient of the deformation procedure since in its absence (61) would still allow solutions for a_0, as it can be observed from (62) in which we set $\bar{a}_1 = 0$. However, it is important to note that in its absence there are *no couplings* of the matter fields to the BF sector. Indeed, the results (48) and $H_k(\delta|d) = H^{\mathrm{BF}}_k(\delta|d)$ for $k = 2, 3, 4$ imply that the earliest step where the matter generators may be brought in during the deformation process is given by the solutions of the "homogeneous" equation $\gamma\bar{a}_1 = 0$ at antighost number one. Since the scope of this paper is to analyze the structure of possible interactions between the matter and the BF fields, in what follows we focus on the conditions that should be satisfied such that $\bar{a}_1$ indeed furnishes a consistent $\bar{a}_0$.

More precisely, we determine the allowed form of the functions $T^{\mu\nu}_{ab}$, $\tilde{T}^{\mu}_{ab}$, T_{ab}, $T^{a}_{\mu b}$ and $\bar{T}^{i}_{b}$ in (60) such that (63) is obeyed. Recalling the definitions (18)–(22), it follows that

$$\delta\bar{a}_1 = -\left(\delta\lambda_b\right)\eta^b. \tag{64}$$

In the meantime, from the definitions (24) and the first two relations in (25) we read that (63) possesses solutions if and only if there exist some bosonic, γ-invariant currents $\bar{j}^{\mu}_{b}$ with both pure ghost and antighost numbers equal to zero,

$$\gamma\bar{j}^{\mu}_{b} = 0,\ \mathrm{pgh}\left(\bar{j}^{\mu}_{b}\right) = 0 = \mathrm{agh}\left(\bar{j}^{\mu}_{b}\right),\ \varepsilon\left(\bar{j}^{\mu}_{b}\right) = 0, \tag{65}$$

such that

$$-\delta\lambda_b = \partial_\mu\bar{j}^{\mu}_{b}. \tag{66}$$

Analyzing the expression of λ_b from (60), after some computation we find that a necessary condition for (66) to hold is

$$T^{\mu\nu}_{ab} = 0,\ \tilde{T}^{\mu}_{ab} = 0,\ T_{ab} = 0. \tag{67}$$

Substituting the partial solutions (67) back into (66), the latter becomes

$$\left(\partial^\mu\varphi_a\right)T^{a}_{\mu b}\left(\left[\omega^{\Delta}\right]\right) + (-)^{\epsilon_i}\frac{\delta^L\mathcal{L}_0}{\delta y^i}\bar{T}^{i}_{b}\left(\left[\omega^{\Delta}\right]\right) = \partial_\mu\bar{j}^{\mu}_{b}. \tag{68}$$

A practical manner of exhibiting solutions to (68), and hence, by virtue of the above discussion, of introducing couplings between the BF and matter theories, is to suppose that there exist some local functions T^{i}_{a} involving only the matter fields and their derivatives, whose Grassmann parities are ε_i, such that

$$(-)^{\varepsilon_i}\frac{\delta^L\mathcal{L}_0}{\delta y^i}T^{i}_{a}\left(\left[y^i\right]\right) = \partial_\mu j^{\mu}_{a}\left(\left[y^i\right]\right). \tag{69}$$

Here, j^{μ}_{a} are some bosonic, local currents that depend only on the purely matter field spectrum. The last relation is nothing but Noether's theorem expressing the appearance of the on-shell conserved currents $j^{\mu}_{a}\left(\left[y^i\right]\right)$ (on-shell means here on the stationary surface of field equations for the purely matter theory) deriving from the invariance of the Lagrangian action of the matter fields under the rigid symmetries

$$\Delta y^i = T^{i}_{a}\left(\left[y^i\right]\right)\xi^a, \tag{70}$$

with ξ^a some constant, bosonic parameters. From now on we work under the hypothesis that the matter theory indeed displays such rigid symmetries.

Equation (69) may be rewritten in terms of the Koszul–Tate differential as

$$\partial_\mu j^{\mu}_{a} = \delta\left(-y^{*}_{i}T^{i}_{a}\right) \equiv \delta\sigma_a, \tag{71}$$

and it correlates the rigid symmetries (70) to certain cohomological classes from the space $H_1(\delta|d)$. Explicitly, it shows that some global symmetries (materialized in the conserved currents j^{μ}_{a}) define some elements σ_a from $H_1(\delta|d)$, i.e., some elements of antighost number equal to one that are δ-closed modulo d. A global symmetry is said to be trivial if the corresponding σ_a are in a trivial class of $H_1(\delta|d)$, hence if they are δ-exact modulo d

$$\sigma_a = \delta\rho_a + \partial_\mu c^{\mu}_{a},\ \mathrm{agh}\left(\rho_a\right) = 2,\ \mathrm{agh}\left(c^{\mu}_{a}\right) = 1. \tag{72}$$

The currents associated with a trivial global symmetry are trivial (see the first reference from [7]), and we assume that this is not the case here.

Inserting (69) in (68), we find that the left-hand side of (68) reduces to a total derivative if and only if

$$\bar{T}^{i}_{b} = T^{i}_{a}\left(\left[y^i\right]\right)U^{a}_{b}(\varphi),\ T^{a}_{\mu b} = j_{c\mu}\left(\left[y^i\right]\right)\frac{\partial U^{c}_{b}(\varphi)}{\partial\varphi_a}, \tag{73}$$

where $U^{a}_{b}(\varphi)$ are some arbitrary functions of the undifferentiated scalar fields. In this case, the bar current from (68) is related to the purely matter one via

$$\bar{j}^{\mu}_{b} = j^{\mu}_{a}\left(\left[y^i\right]\right)U^{a}_{b}(\varphi). \tag{74}$$

Using the solutions (67) and (73) in (60), we completely determine $\bar{a}_1$ in the form

$$\bar{a}_1 = y^{*}_{i}T^{i}_{a}\left(\left[y^i\right]\right)U^{a}_{b}(\varphi)\eta^b + H^{*\mu}_{a}j_{c\mu}\left(\left[y^i\right]\right)\frac{\partial U^{c}_{b}(\varphi)}{\partial\varphi_a}\eta^b. \tag{75}$$

With $\bar{a}_1$ at hand, from (60) we find that the solution to (63) is

$$\bar{a}_0 = j^{\mu}_{a}\left(\left[y^i\right]\right)U^{a}_{b}(\varphi)A^{b}_{\mu}, \tag{76}$$

which, correlated with (62), enables us to write the full antighost number zero component in the first-order deformation like

$$\begin{aligned} a_0 = {} & j^{\mu}_{a}\left(\left[y^i\right]\right)U^{a}_{b}(\varphi)A^{b}_{\mu} - W_{ab}(\varphi)A^{a\mu}H^{b}_{\mu} \\ & + \frac{1}{2}M^{c}_{ab}(\varphi)A^{a}_{\mu}A^{b}_{\nu}B^{\mu\nu}_{c} \\ & + \frac{1}{2}\varepsilon^{\mu\nu\rho\lambda}\left(M^{ab}(\varphi)B_{a\mu\nu}B_{b\rho\lambda}\right. \\ & \left. - \frac{1}{2\cdot 4!}M_{abcd}(\varphi)A^{a}_{\mu}A^{b}_{\nu}A^{c}_{\rho}A^{d}_{\lambda}\right). \end{aligned} \tag{77}$$

The basic conclusion of the above discussion is that the appearance of consistent couplings (at order one in the deformation parameter) of the matter fields to the BF ones is obtained under the hypothesis that the matter theory is invariant under some (non-trivial) bosonic global transformations of the type (70), that result, via Noether's theorem (69), in the (non-trivial) conserved currents j^{μ}_{a}. It is essential that the number of rigid symmetries is equal to the number of BF fields from the collection (N).

Putting together the formulas (54), (55), (57), (59), (75) and (77), we conclude that the first-order deformation of the solution to the master equation for the model under study can be written in the form

$$S_1 = \int d^4x\left(\left(\frac{\partial W_{ab}}{\partial\varphi_c}C^{*\mu\nu\rho\lambda}_{c}\right.\right. \\ + \frac{\partial^2 W_{ab}}{\partial\varphi_c\partial\varphi_d}\left(H^{*[\mu}_{c}C^{*\nu\rho\lambda]}_{d} + C^{*[\mu\nu}_{c}C^{*\rho\lambda]}_{d}\right)$$

$$+ \frac{\partial^3 W_{ab}}{\partial\varphi_c \partial\varphi_d \partial\varphi_e} H_c^{*[\mu} H_d^{*\nu} C_e^{*\rho\lambda]}$$

$$+ \frac{\partial^4 W_{ab}}{\partial\varphi_c \partial\varphi_d \partial\varphi_e \partial\varphi_f} H_c^{*\mu} H_d^{*\nu} H_e^{*\rho} H_f^{*\lambda} \Bigg) \eta^a C^b_{\mu\nu\rho\lambda}$$

$$- \frac{1}{4} \Bigg(\frac{\partial M^c_{ab}}{\partial\varphi_d} C_d^{*\mu\nu\rho\lambda}$$

$$+ \frac{\partial^2 M^c_{ab}}{\partial\varphi_d \partial\varphi_e} \left(H_d^{*[\mu} C_e^{*\nu\rho\lambda]} + C_d^{*[\mu\nu} C_e^{*\rho\lambda]} \right)$$

$$+ \frac{\partial^3 M^c_{ab}}{\partial\varphi_d \partial\varphi_e \partial\varphi_f} H_d^{*[\mu} H_e^{*\nu} C_f^{*\rho\lambda]}$$

$$+ \frac{\partial^4 M^c_{ab}}{\partial\varphi_d \partial\varphi_e \partial\varphi_f \partial\varphi_g} H_d^{*\mu} H_e^{*\nu} H_f^{*\rho} H_g^{*\lambda} \Bigg) \eta^a \eta^b \eta_{c\mu\nu\rho\lambda}$$

$$+ \frac{1}{2} \varepsilon_{\mu\nu\rho\lambda} \Bigg(\Bigg(\frac{\partial M^{ab}}{\partial\varphi_c} C_c^{*\mu\nu\rho\lambda}$$

$$+ \frac{\partial^2 M^{ab}}{\partial\varphi_c \partial\varphi_d} \left(H_c^{*[\mu} C_d^{*\nu\rho\lambda]} + C_c^{*[\mu\nu} C_d^{*\rho\lambda]} \right)$$

$$+ \frac{\partial^3 M^{ab}}{\partial\varphi_c \partial\varphi_d \partial\varphi_e} H_c^{*[\mu} H_d^{*\nu} C_e^{*\rho\lambda]}$$

$$+ \frac{\partial^4 M^{ab}}{\partial\varphi_c \partial\varphi_d \partial\varphi_e \partial\varphi_f} H_c^{*\mu} H_d^{*\nu} H_e^{*\rho} H_f^{*\lambda} \Bigg) \eta_a^{\sigma\tau\kappa\varsigma} \eta_{b\sigma\tau\kappa\varsigma}$$

$$- \frac{1}{2 \cdot (4!)^2} \Bigg(\frac{\partial M_{mnpq}}{\partial\varphi_c} C_c^{*\mu\nu\rho\lambda}$$

$$+ \frac{\partial^2 M_{mnpq}}{\partial\varphi_c \partial\varphi_d} \left(H_c^{*[\mu} C_d^{*\nu\rho\lambda]} + C_c^{*[\mu\nu} C_d^{*\rho\lambda]} \right)$$

$$+ \frac{\partial^3 M_{mnpq}}{\partial\varphi_c \partial\varphi_d \partial\varphi_e} H_c^{*[\mu} H_d^{*\nu} C_e^{*\rho\lambda]}$$

$$+ \frac{\partial^4 M_{mnpq}}{\partial\varphi_c \partial\varphi_d \partial\varphi_e \partial\varphi_f} H_c^{*\mu} H_d^{*\nu} H_e^{*\rho} H_f^{*\lambda} \Bigg) \eta^m \eta^n \eta^p \eta^q \Bigg)$$

$$+ \Bigg(\frac{\partial W_{ab}}{\partial\varphi_c} C_c^{*\mu\nu\rho} + \frac{\partial^2 W_{ab}}{\partial\varphi_c \partial\varphi_d} H_c^{*[\mu} C_d^{*\nu\rho]}$$

$$+ \frac{\partial^3 W_{ab}}{\partial\varphi_c \partial\varphi_d \partial\varphi_e} H_c^{*\mu} H_d^{*\nu} H_e^{*\rho} \Bigg)$$

$$\times \left(-\eta^a C^b_{\mu\nu\rho} + 4 A^{a\lambda} C^b_{\mu\nu\rho\lambda} \right)$$

$$+ 2 \Bigg(6 \left(\frac{\partial W_{ab}}{\partial\varphi_c} C_c^{*\mu\nu} + \frac{\partial^2 W_{ab}}{\partial\varphi_c \partial\varphi_d} H_c^{*\mu} H_d^{*\nu} \right) B^{*a\rho\lambda}$$

$$+ 4 \frac{\partial W_{ab}}{\partial\varphi_c} H_c^{*\mu} \eta^{*a\nu\rho\lambda} + W_{ab} \eta^{*a\mu\nu\rho\lambda} \Bigg) C^b_{\mu\nu\rho\lambda}$$

$$+ \frac{1}{2} \Bigg(\frac{\partial M^c_{ab}}{\partial\varphi_d} C_d^{*\mu\nu\rho}$$

$$+ \frac{\partial^2 M^c_{ab}}{\partial\varphi_d \partial\varphi_e} H_d^{*[\mu} C_e^{*\nu\rho]} + \frac{\partial^3 M^c_{ab}}{\partial\varphi_d \partial\varphi_e \partial\varphi_f} H_d^{*\mu} H_e^{*\nu} H_f^{*\rho} \Bigg)$$

$$\times \left(\frac{1}{2} \eta^a \eta^b \eta_{c\mu\nu\rho} - 4 A^{a\lambda} \eta^b \eta_{c\mu\nu\rho\lambda} \right)$$

$$- \Bigg(6 \left(\frac{\partial M^c_{ab}}{\partial\varphi_d} C_d^{*\mu\nu} + \frac{\partial^2 M^c_{ab}}{\partial\varphi_d \partial\varphi_e} H_d^{*\mu} H_e^{*\nu} \right) B^{*a\rho\lambda}$$

$$+ 4 \frac{\partial M^c_{ab}}{\partial\varphi_d} H_d^{*\mu} \eta^{*a\nu\rho\lambda} + M^c_{ab} \eta^{*a\mu\nu\rho\lambda} \Bigg) \eta^b \eta_{c\mu\nu\rho\lambda}$$

$$- \varepsilon_{\mu\nu\rho\lambda} \Bigg(\frac{\partial M^{ab}}{\partial\varphi_c} C_c^{*\sigma\tau\kappa} + \frac{\partial^2 M^{ab}}{\partial\varphi_c \partial\varphi_d} H_c^{*[\sigma} C_d^{*\tau\kappa]}$$

$$+ \frac{\partial^3 M^{ab}}{\partial\varphi_c \partial\varphi_d \partial\varphi_e} H_c^{*\sigma} H_d^{*\tau} H_e^{*\kappa} \Bigg) \eta_{a\sigma\tau\kappa} \eta_b^{\mu\nu\rho\lambda}$$

$$- \frac{\varepsilon_{\mu\nu\rho\lambda}}{(4!)^2} \Bigg(4 \Bigg(\frac{\partial M_{mnpq}}{\partial\varphi_c} C_c^{*\mu\nu\rho} + \frac{\partial^2 M_{mnpq}}{\partial\varphi_c \partial\varphi_d} H_c^{*[\mu} C_d^{*\nu\rho]}$$

$$+ \frac{\partial^3 M_{mnpq}}{\partial\varphi_c \partial\varphi_d \partial\varphi_e} H_c^{*\mu} H_d^{*\nu} H_e^{*\rho} \Bigg) A^{m\lambda}$$

$$+ 12 \left(\frac{\partial M_{mnpq}}{\partial\varphi_c} C_c^{*\mu\nu} + \frac{\partial^2 M_{mnpq}}{\partial\varphi_c \partial\varphi_d} H_c^{*\mu} H_d^{*\nu} \right) B^{*m\rho\lambda}$$

$$+ 8 \frac{\partial M_{mnpq}}{\partial\varphi_c} H_c^{*\mu} \eta^{*m\nu\rho\lambda} + 2 M_{mnpq} \eta^{*m\mu\nu\rho\lambda} \Bigg) \eta^n \eta^p \eta^q$$

$$+ \left(\frac{\partial W_{ab}}{\partial\varphi_c} C_c^{*\mu\nu} + \frac{\partial^2 W_{ab}}{\partial\varphi_c \partial\varphi_d} H_c^{*\mu} H_d^{*\nu} \right)$$

$$\times \left(\eta^a C^b_{\mu\nu} - 3 A^{a\rho} C^b_{\mu\nu\rho} \right)$$

$$- 2 \left(3 \frac{\partial W_{ab}}{\partial\varphi_c} H_c^{*\mu} B^{*a\nu\rho} + W_{ab} \eta^{*a\mu\nu\rho} \right) C^b_{\mu\nu\rho}$$

$$- \frac{1}{2} \left(\frac{\partial M^c_{ab}}{\partial\varphi_d} C_d^{*\mu\nu} + \frac{\partial^2 M^c_{ab}}{\partial\varphi_d \partial\varphi_e} H_d^{*\mu} H_e^{*\nu} \right)$$

$$\times \left(\frac{1}{2} \eta^a \eta^b B_{c\mu\nu} - 3 A^{a\rho} \eta^b \eta_{c\mu\nu\rho} \right)$$

$$+ \left(3 \frac{\partial M^c_{ab}}{\partial\varphi_d} H_d^{*\mu} B^{*a\nu\rho} + M^c_{ab} \eta^{*a\mu\nu\rho} \right) \eta^b \eta_{c\mu\nu\rho}$$

$$+ \frac{1}{2} \left(- \frac{\partial M^c_{ab}}{\partial\varphi_d} H_d^{*\mu} A^*_{c\mu} + M^c_{ab} \eta^*_c \right) \eta^a \eta^b$$

$$+ \Bigg(3 \left(\frac{\partial M^c_{ab}}{\partial\varphi_d} C_d^{*\mu\nu} + \frac{\partial^2 M^c_{ab}}{\partial\varphi_d \partial\varphi_e} H_d^{*\mu} H_e^{*\nu} \right) A^{a\rho}$$

$$+ 12 \frac{\partial M^c_{ab}}{\partial\varphi_d} H_d^{*\mu} B^{*a\nu\rho} + 4 M^c_{ab} \eta^{*a\mu\nu\rho} \Big) A^{b\lambda} \eta_{c\mu\nu\rho\lambda}$$

$$- 6 M^c_{ab} B^{*a}_{\mu\nu} B^{*b}_{\rho\lambda} \eta_c^{\mu\nu\rho\lambda}$$

$$+ \frac{9}{2} \varepsilon^{\mu\nu\rho\lambda} \left(\frac{\partial M^{ab}}{\partial\varphi_d} C^*_{d\mu\nu} + \frac{\partial^2 M^{ab}}{\partial\varphi_d \partial\varphi_e} H^*_{d\mu} H^*_{e\nu} \right) \eta_{a\rho\sigma\tau} \eta_{b\lambda}{}^{\sigma\tau}$$

$$+ \varepsilon_{\mu\nu\rho\lambda} \Bigg(\left(\frac{\partial M^{ab}}{\partial\varphi_d} C_d^{*\sigma\tau} + \frac{\partial^2 M^{ab}}{\partial\varphi_d \partial\varphi_e} H_d^{*\sigma} H_e^{*\tau} \right) B_{b\sigma\tau}$$

$$+ 2 \frac{\partial M^{ab}}{\partial\varphi_d} H_d^{*\sigma} A^*_{b\sigma} - 2 M^{ab} \eta^*_b \Bigg) \eta_a^{\mu\nu\rho\lambda}$$

$$+ \frac{3\varepsilon_{\mu\nu\rho\lambda}}{(4!)^2} \Bigg(6 \left(\frac{\partial M_{mnpq}}{\partial\varphi_d} C_d^{*\mu\nu} + \frac{\partial^2 M_{mnpq}}{\partial\varphi_d \partial\varphi_e} H_d^{*\mu} H_e^{*\nu} \right)$$

$$\times A^{m\rho} A^{n\lambda}$$

$$+ 24 \frac{\partial M_{mnpq}}{\partial\varphi_d} H_d^{*\mu} B^{*m\nu\rho} A^{n\lambda}$$

$$+ 8 M_{mnpq} \eta^{*m\mu\nu\rho} A^{n\lambda} - 12 M_{mnpq} B^{*m\mu\nu} B^{*n\rho\lambda} \Big) \eta^p \eta^q$$

$$+ \frac{\partial W_{ab}}{\partial\varphi_c} H_c^{*\mu} \left(-\eta^a H^b_\mu + 2 A^{a\nu} C^b_{\mu\nu} \right)$$

$$+ W_{ab} \left(2 B^{*a}_{\mu\nu} C^{b\mu\nu} - \varphi^{*a} \eta^b \right)$$

$$- \frac{\partial M^c_{ab}}{\partial\varphi_d} H_d^{*\mu} A^{a\nu} \left(\eta^b B_{c\mu\nu} + \frac{3}{2} A^{b\rho} \eta_{c\mu\nu\rho} \right)$$

$$
\begin{aligned}
&- M^c_{ab}\left(B^{*a}_{\mu\nu}\eta^b B^{\mu\nu}_c + A^a_\mu \eta^b A^{*\mu}_c + 3B^{*a}_{\mu\nu}A^b_\rho \eta^{\mu\nu\rho}_c\right)\\
&+ 2\varepsilon_{\nu\rho\sigma\tau}\left(\frac{\partial M^{ab}}{\partial\varphi_c}H^*_{c\mu}B^{\mu\nu}_a - M^{ab}A^{*\nu}_a\right)\eta^{\rho\sigma\tau}_b\\
&+ \frac{\varepsilon^{\mu\nu\rho\lambda}}{4!}\left(\frac{\partial M_{mnpq}}{\partial\varphi_c}H^*_{c\mu}A^m_\nu + 3M_{mnpq}B^{*m}_{\mu\nu}\right)A^n_\rho A^p_\lambda \eta^q\\
&+ y^*_i T^i_a U^a_b \eta^b + H^{*\mu}_m j_{a\mu}\frac{\partial U^a_b}{\partial\varphi_m}\eta^b - W_{ab}A^{a\mu}H^b_\mu\\
&+ \frac{1}{2}M^c_{ab}A^a_\mu A^b_\nu B^{\mu\nu}_c + j^\mu_a U^a_b A^b_\mu \qquad (78)\\
&+ \frac{\varepsilon^{\mu\nu\rho\lambda}}{2}\left(M^{ab}B_{a\mu\nu}B_{b\rho\lambda} - \frac{1}{2\cdot 4!}M_{abcd}A^a_\mu A^b_\nu A^c_\rho A^d_\lambda\right)\Bigg).
\end{aligned}
$$

It is by construction a s-cocycle of ghost number zero, such that $S + gS_1$ is solution to the master equation (31) up to order g.

4.2 Higher-order deformations

It is clear from (78) that S_1 strongly depends on the structure of the matter theory. The only (reasonable) assumptions made so far on the "free" matter theory are that its Lagrangian density $\mathcal{L}_0\left(\left[y^i\right]\right)$ is at most second order in the derivatives of y^i and that it separately describes a "normal" theory of Cauchy order equal to one. Until now we did not restrict in any way the derivative order of the interacting Lagrangian density, given at the first order in the coupling constant by a_0 as in (77). However, as announced in the beginning of this section, we will ask that the interactions preserve the differential order with respect to the "free" field equations. Given the first-order differential behavior of the BF field equations resulting from the "free" action (1), it follows that each term in a_0 must be restricted to have at most one spacetime derivative. The quantities in a_0 that may contain spacetime derivatives of the fields are proportional to $j^\mu_a\left(\left[y^i\right]\right)$, and hence we must ask that the conserved matter currents have no more than one derivative. In view of Noether's theorem (69) it is then sufficient to take the generators T^i_a of the rigid symmetries (70) to be polynomials in the undifferentiated matter fields (or even infinite series in the subset of commuting such fields). In order to fix the ideas and manifestly ensure that the Grassmann parity of T^i_a is ε_i from now on we consider the case where the generators of the rigid symmetries (70) are linear in the matter fields, i.e.

$$
T^i_a = \left(T_a\right)^i_{\ j}\, y^j, \qquad (79)
$$

with $\left(T_a\right)^i_{\ j}$ denoting the components of some constant matrices T_a. Consequently, we obtain that the derivative order of j^μ_a is less than that of the matter Lagrangian density by one unit, namely it may be either zero or one.

Next, we discuss the higher-order deformation equations (33), (34) etc. Initially, we analyze under what conditions the first-order deformation (78) is consistent at the second order in the coupling constant, namely (33) holds. We will see that these conditions impose various restrictions on the functions entering (78), so on the one hand they fix the expression of S_1 itself and, on the other hand, allow us to predict whether non-trivial second- and possibly higher-order deformations of the solution to the master equation appear[3]. The second-order deformation is governed by (33), which, if we maintain the notation from (30) and consider that $(S_1, S_1) = \int \mathrm{d}^4x\, \Delta$, takes the local form

$$
\Delta = -\frac{1}{2}sb + \partial^\mu\theta_\mu. \qquad (80)
$$

At this point it is necessary to make some specifications. It is clear that the expression of Δ depends, beside the BF sector, also on the (derivative) structure of the corresponding matter currents j^μ_a. This is why we will approach distinctly the situation where the matter currents display no derivatives from the case where the derivative order of these currents is equal to one.

Assuming that the matter currents have no derivatives (or, equivalently, that the matter Lagrangian density is first-order derivative), with the help of (78) in which we set (79) we infer that

$$
\begin{aligned}
\Delta &= K^{abc}t_{abc} + K^{abc}_d\frac{\partial t_{abc}}{\partial\varphi_d} + K^{abc}_{de}\frac{\partial^2 t_{abc}}{\partial\varphi_d\partial\varphi_e}\\
&+ K^{abc}_{def}\frac{\partial^3 t_{abc}}{\partial\varphi_d\partial\varphi_e\partial\varphi_f} + K^{abc}_{defg}\frac{\partial^4 t_{abc}}{\partial\varphi_d\partial\varphi_e\partial\varphi_f\partial\varphi_g} + U^{abc}_d t^d_{abc}\\
&+ U^{abc}_{d,e}\frac{\partial t^d_{abc}}{\partial\varphi_e} + U^{abc}_{d,ef}\frac{\partial^2 t^d_{abc}}{\partial\varphi_e\partial\varphi_f} + U^{abc}_{d,efg}\frac{\partial^3 t^d_{abc}}{\partial\varphi_e\partial\varphi_f\partial\varphi_g}\\
&+ U^{abc}_{d,efgh}\frac{\partial^4 t^d_{abc}}{\partial\varphi_e\partial\varphi_f\partial\varphi_g\partial\varphi_h} + K^{abcdf}t_{abcdf}\\
&+ K^{abcdf}_e\frac{\partial t_{abcdf}}{\partial\varphi_e} + K^{abcdf}_{eg}\frac{\partial^2 t_{abcdf}}{\partial\varphi_e\partial\varphi_g} + K^{abcdf}_{egh}\frac{\partial^3 t_{abcdf}}{\partial\varphi_e\partial\varphi_g\partial\varphi_h}\\
&+ K^{abcdf}_{eghl}\frac{\partial^4 t_{abcdf}}{\partial\varphi_e\partial\varphi_g\partial\varphi_h\partial\varphi_l} + K^a_b t^b_a + K^a_{b,c}\frac{\partial t^b_a}{\partial\varphi_c}\\
&+ K^a_{b,cd}\frac{\partial^2 t^b_a}{\partial\varphi_c\partial\varphi_d} + K^a_{b,cde}\frac{\partial^3 t^b_a}{\partial\varphi_c\partial\varphi_d\partial\varphi_e}\\
&+ K^a_{b,cdef}\frac{\partial^4 t^b_a}{\partial\varphi_c\partial\varphi_d\partial\varphi_e\partial\varphi_f} + K^c_{ab}t^{ab}_c + K^c_{ab,d}\frac{\partial t^{ab}_c}{\partial\varphi_d}\\
&+ K^c_{ab,de}\frac{\partial^2 t^{ab}_c}{\partial\varphi_d\partial\varphi_e} + K^c_{ab,def}\frac{\partial^3 t^{ab}_c}{\partial\varphi_d\partial\varphi_e\partial\varphi_f}\\
&+ K^c_{ab,defg}\frac{\partial^4 t^{ab}_c}{\partial\varphi_d\partial\varphi_e\partial\varphi_f\partial\varphi_g} + \bar{\Delta}\\
&\equiv \Pi + \bar{\Delta}, \qquad (81)
\end{aligned}
$$

where $\bar{\Delta}$ is responsible for the occurrence of the matter sector and its expression is

$$
\bar{\Delta} = -4\varepsilon_{\mu\nu\rho\lambda}
$$

[3] Strictly speaking, we should have added to (78) also the solutions to the equation $\gamma a_0' = \partial_\mu \tau^\mu$ at antighost number zero. For the lack of simplicity we have omitted such solutions since their consistency can be shown to enforce their triviality, independently of the consistency equation for (78), which is further discussed.

$$\times \left(\left(\left(H_m^{*\sigma} j_{a\sigma} \frac{\partial \left(U_e^a M^{eb} \right)}{\partial \varphi_m} + y_i^* \left(T_a \right)^i{}_j y^j U_e^a M^{eb} \right) \eta_b^{\mu\nu\rho\lambda} + j_a^\mu U_e^a M^{eb} \eta_b^{\nu\rho\lambda} \right) + y_i^* \left(\left(T_a \right)^i{}_j \left(U_e^a \frac{\partial W_{bc}}{\partial \varphi_e} + W_{eb} \frac{\partial U_c^a}{\partial \varphi_e} - W_{ec} \frac{\partial U_b^a}{\partial \varphi_e} \right) + \left[T_d, T_e \right]^i{}_j U_b^d U_c^e \right) y^j \eta^b \eta^c + H_{m\mu}^* \left(j_a^\mu \frac{\partial}{\partial \varphi_m} \left(U_e^a \frac{\partial W_{bc}}{\partial \varphi_e} + W_{eb} \frac{\partial U_c^a}{\partial \varphi_e} - W_{ec} \frac{\partial U_b^a}{\partial \varphi_e} \right) + \frac{\delta^R j_a^\mu}{\delta y^i} \left(T_e \right)^i{}_j y^j \left(\frac{\partial U_b^a}{\partial \varphi_m} U_c^e - \frac{\partial U_c^a}{\partial \varphi_m} U_b^e \right) \right) \eta^b \eta^c + 2 \left(j_a^\mu \left(U_e^a \frac{\partial W_{bc}}{\partial \varphi_e} + W_{eb} \frac{\partial U_c^a}{\partial \varphi_e} - W_{ec} \frac{\partial U_b^a}{\partial \varphi_e} \right) + \frac{\delta^R j_a^\mu}{\delta y^i} \left(T_e \right)^i{}_j y^j U_c^e U_b^a \right) A_\mu^b \eta^c. \tag{82}$$

The form of the t's involved in (81) reads

$$t_{abc} = W_{ec} M_{ab}^c + W_{ea} \frac{\partial W_{bc}}{\partial \varphi_e} + W_{eb} \frac{\partial W_{ca}}{\partial \varphi_e}, \tag{83}$$

$$t_{abc}^d = W_{e[a} \frac{\partial M_{bc]}^d}{\partial \varphi_e} + M_{e[a}^d M_{bc]}^e + M^{de} M_{eabc}, \tag{84}$$

$$t_{abcdf} = W_{e[a} \frac{\partial M_{bcdf]}}{\partial \varphi_e} + M_{e[abc} M_{df]}^e, \tag{85}$$

$$t_a^b = M^{be} W_{ea}, \tag{86}$$

$$t_a^{bc} = W_{ea} \frac{\partial M^{bc}}{\partial \varphi_e} + M_{ea}^{(b} M^{c)e}, \tag{87}$$

while the remaining objects, of the type K and U, can be found in Appendix 6. The notation $(bc\cdots)$ signifies complete symmetrization with respect to the indices between parentheses such as to include all the independent terms only once and without normalization factors. We recall that the expression of $\bar{\Delta}$ from (82) was obtained under the assumption that the currents j_a^μ exhibit *no derivatives.*

In the complementary situation, where the matter currents do contain derivatives (which is the same as assuming that the matter Lagrangian density is second-order derivative), we find that the non-integrated density of $(S_1, S_1) = \int \mathrm{d}^4 x \, \Delta'$ can be written as

$$\Delta' = \Delta + \Lambda \equiv \Pi + \bar{\Delta} + \Lambda, \tag{88}$$

where Δ is given in (81) and Λ contains derivatives, involves both the BF and matter sectors and depends on the concrete form of j_a^μ. The general expression of Λ is nevertheless not illuminating in the sequel. Since Δ' includes Δ, it is enough to analyze the consistency of the first-order deformation using directly the former quantity. However, when relevant differences between the cases where the currents do or do not contain derivatives arise, they will be clearly emphasized. There are two main types of terms in the right-hand side of (88):
(i) the first kind involves only the BF sector and was generically denoted by Π;
(ii) the second variety combines the matter and the BF spectra and is designated by $\bar{\Delta} + \Lambda$, with $\bar{\Delta}$ given in (82). Due to their different nature, Π and $\bar{\Delta} + \Lambda$ must separately be s-boundaries modulo d, i.e. each of them has to be written in a form similar to the right-hand side of (80).

None of the elements of the type (i) can be written as in the right-hand side of (80) because none of them contains spacetime derivatives, as does the action of s on all fields, ghosts and antifields from the BF sector. Thereby, t_{abc}, t_{abc}^d, t_{abcdf}, t_a^b and t_a^{bc} must vanish:

$$t_{abc} = 0,\ t_{abc}^d = 0,\ t_{abcdf} = 0,\ t_a^b = 0,\ t_a^{bc} = 0. \tag{89}$$

Using the expressions (83)–(87), we see that the solution to (89) is

$$M_{ab}^c = \frac{\partial W_{ab}}{\partial \varphi_c},\ M_{abcd} = f_{e[ab} \frac{\partial W_{cd]}}{\partial \varphi_e},\ M^{ab} = 0, \tag{90}$$

where the functions W_{ab} (which are antisymmetric now due to the first relation in (90) and to the established antisymmetry of M_{ab}^c) are restricted to satisfy the identity

$$W_{e[a} \frac{\partial W_{bc]}}{\partial \varphi_e} = 0, \tag{91}$$

and f_{abc} are arbitrary constants, completely antisymmetric in their indices.

Because of (90), we find that Δ' given by (88) reduces only to components of the kind (ii):

$$\Delta' = y_i^* \left(\left(T_a \right)^i{}_j \left(U_e^a \frac{\partial W_{bc}}{\partial \varphi_e} + W_{eb} \frac{\partial U_c^a}{\partial \varphi_e} - W_{ec} \frac{\partial U_b^a}{\partial \varphi_e} \right) + \left[T_d, T_e \right]^i{}_j U_b^d U_c^e \right) y^j \eta^b \eta^c + H_{m\mu}^* \left(j_a^\mu \frac{\partial}{\partial \varphi_m} \left(U_e^a \frac{\partial W_{bc}}{\partial \varphi_e} + W_{eb} \frac{\partial U_c^a}{\partial \varphi_e} - W_{ec} \frac{\partial U_b^a}{\partial \varphi_e} \right) + \frac{\delta^R j_a^\mu}{\delta y^i} \left(T_e \right)^i{}_j y^j \left(\frac{\partial U_b^a}{\partial \varphi_m} U_c^e - \frac{\partial U_c^a}{\partial \varphi_m} U_b^e \right) \right) \eta^b \eta^c + 2 \left(j_a^\mu \left(U_e^a \frac{\partial W_{bc}}{\partial \varphi_e} + W_{eb} \frac{\partial U_c^a}{\partial \varphi_e} - W_{ec} \frac{\partial U_b^a}{\partial \varphi_e} \right) + \frac{\delta^R j_a^\mu}{\delta y^i} \left(T_e \right)^i{}_j y^j U_c^e U_b^a \right) A_\mu^b \eta^c + \Lambda. \tag{92}$$

We observe that the term from (92) proportional to y_i^* cannot be written in a s-exact modulo d form for the same reason as before. In view of this, we impose that

$$\left(U_e^a \frac{\partial W_{bc}}{\partial \varphi_e} + W_{eb} \frac{\partial U_c^a}{\partial \varphi_e} - W_{ec} \frac{\partial U_b^a}{\partial \varphi_e} \right) \left(T_a \right)^i{}_j + \left[T_d, T_e \right]^i{}_j U_b^d U_c^e = 0. \tag{93}$$

There are two kinds of solutions to the system (93). The first one (to be called "type I solution") is

$$U_b^a = k^{ae} W_{eb},\ \left[T_a, T_b \right] = 0, \tag{94}$$

and it does not impose further constraints on the functions W_{ab}, but merely restricts the matrices T_a to be commutative (the second relation in (94)). In (94) k^{ae} are some constants. Due to the identities (91), the antisymmetric functions W_{ab} can be viewed as the two-tensor on a Poisson manifold with the dynamical scalar fields taken as local coordinates on the target space. The second one (to be named "type II solution") reads

$$W_{ab} = \bar{f}^{c}_{ab}\varphi_c + F_{ab}, \tag{95}$$
$$M^{a}_{bc} = \bar{f}^{a}_{bc}, \tag{96}$$
$$U^{a}_{b} = \delta^{a}_{b}, \tag{97}$$
$$[T_a, T_b] = -\bar{f}^{c}_{ab}T_c, \tag{98}$$

and it restricts the functions W_{ab} to be at most linear in the scalar fields. In the above, $\bar{f}^{c}_{ab}$ and F_{ab} are some constants, antisymmetric in their lower indices. As a consequence of the identities (91), we find that these constants are further subject to the conditions

$$\bar{f}^{d}_{e[a}\bar{f}^{e}_{bc]} = 0,\ F_{e[a}\bar{f}^{e}_{bc]} = 0. \tag{99}$$

We can thus interpret $\bar{f}^{c}_{ab}$ like the structure constants of a certain Lie algebra $L(\mathcal{G})$, where by $\mathcal{G}$ we denoted the unique (since the Lie algebra is by hypothesis finite-dimensional) connected, simply-connected Lie group having this algebra as its Lie algebra. Then, according to (98), the matrices T_a of elements $(T_a)^{i}_{\ k}$ can be viewed as a basis of infinitesimal generators of an arbitrary linear representation of dimension I (the number of matter fields) of $L(\mathcal{G})$. With these two types of solutions at hand, in the sequel we analyze the existence of higher-order deformations of the solution to the master equation.

4.2.1 Type I solutions

Substituting (94) in (92) we obtain

$$(S_1, S_1) = \int \mathrm{d}^4x \left(2\frac{\delta^R j^{\mu}_{a}}{\delta y^i}(T_b)^{i}_{\ j}y^j k^{am}k^{bn}W_{nc}\eta^c \times \left(W_{md}A^{d}_{\mu} + H^{*}_{e\mu}\frac{\partial W_{md}}{\partial\varphi_e}\eta^d\right) + \Lambda_{\mathrm{I}}\right), \tag{100}$$

where Λ_{I} means Λ restricted to the type I solutions. Two major situations met in practical applications deserve special attention.

I.a. Initially, we consider the case where the matter currents are invariant under the gauge version of the genuine rigid symmetries (70),

$$\frac{\delta^R j^{\mu}_{a}}{\delta y^i}(T_b)^{i}_{\ j}y^j = 0. \tag{101}$$

It is obvious that if the matter currents j^{μ}_{a} contain derivatives, then they cannot be invariant under the gauge version of the rigid symmetries (70). As a consequence, the formula (101) might hold only if these currents involve no derivatives, in which situation Λ_{I} vanishes. Then, from (100) we infer that $(S_1, S_1) = 0$, and so we can take $S_2 = 0$, and, in fact, $S_k = 0$ for all $k \geq 2$. As a consequence, the deformed solution to the master equation that is *consistent to all orders in the coupling constant* results in this case from (78) where we set (79), (90) and (94), and reads

$$\begin{aligned}
\bar{S}^{(\mathrm{I.a})} = \int \mathrm{d}^4x \Bigg(& H^{a}_{\mu}\hat{D}^{\mu}\varphi_a + \frac{1}{2}B^{\mu\nu}_{a}\hat{F}^{a}_{\mu\nu} \\
& - \frac{g}{4\cdot 4!}\varepsilon^{\mu\nu\rho\lambda}f_{m[ab}\frac{\partial W_{cd]}}{\partial\varphi_m}A^{a}_{\mu}A^{b}_{\nu}A^{c}_{\rho}A^{d}_{\lambda} \\
& + \mathcal{L}_0\left([y^i]\right) + g j^{\mu}_{a}k^{ae}W_{eb}A^{b}_{\mu} \\
& + H^{*\mu}_{a}\Bigg(2\left(\hat{D}^{\nu}\right)^{a}_{\ b}C^{b}_{\mu\nu} + g j_{m\mu}k^{me}\frac{\partial W_{eb}}{\partial\varphi_a}\eta^b \\
& - g\frac{\partial W_{bc}}{\partial\varphi_a}\eta^b H^{c}_{\mu} - g\frac{\partial^2 W_{bd}}{\partial\varphi_a\partial\varphi_c}A^{b\nu}\left(\eta^d B_{c\mu\nu} + \frac{3}{2}A^{d\rho}\eta_{c\mu\nu\rho}\right) \\
& + g\frac{\varepsilon_{\mu\nu\rho\lambda}}{4!}f_{c[mn}\frac{\partial^2 W_{pq]}}{\partial\varphi_a\partial\varphi_c}A^{m\nu}A^{n\rho}A^{p\lambda}\eta^q\Bigg) \\
& - B^{*a}_{\mu\nu}\Bigg(3\left(\hat{D}_{\rho}\right)^{\ b}_{a}\eta^{\mu\nu\rho}_{b} - 2gW_{ab}C^{b\mu\nu} \\
& + g\frac{\partial W_{ab}}{\partial\varphi_c}\eta^b B^{\mu\nu}_{c} - \frac{g}{8}\varepsilon^{\mu\nu\rho\lambda}f_{d[ab}\frac{\partial W_{pq]}}{\partial\varphi_d}A^{b}_{\rho}A^{p}_{\lambda}\eta^q\Bigg) \\
& + A^{*\mu}_{a}\left(\hat{D}_{\mu}\right)^{a}_{\ b}\eta^b - g\varphi^{*a}W_{ab}\eta^b \\
& + g y^{*}_{i}(T_a)^{i}_{\ j}y^j k^{ae}W_{eb}\eta^b \\
& - C^{*\mu\nu}_{a}\Bigg(3\left(\hat{D}^{\rho}\right)^{a}_{\ b}C^{b}_{\mu\nu\rho} - g\frac{\partial W_{bc}}{\partial\varphi_a}\eta^b C^{c}_{\mu\nu} \\
& - g\frac{\partial^2 W_{bc}}{\partial\varphi_a\partial\varphi_d}\Big(3A^{b\rho}A^{c\lambda}\eta_{d\mu\nu\rho\lambda} \\
& - \left(\frac{1}{4}\eta^b B_{d\mu\nu} + \frac{3}{2}A^{b\rho}\eta_{d\mu\nu\rho}\right)\eta^c\Big) \\
& - \frac{3g}{4\cdot 4!}\varepsilon_{\mu\nu\rho\lambda}f_{d[mn}\frac{\partial^2 W_{pq]}}{\partial\varphi_a\partial\varphi_d}A^{m\rho}A^{n\lambda}\eta^p\eta^q\Bigg) \\
& + \eta^{*a}_{\mu\nu\rho}\Bigg(4\left(\hat{D}_{\lambda}\right)^{\ b}_{a}\eta^{\mu\nu\rho\lambda}_{b} - 2gW_{ab}C^{b\mu\nu\rho} + g\frac{\partial W_{ab}}{\partial\varphi_c}\eta^b\eta^{\mu\nu\rho}_{c} \\
& + \frac{g}{4!}\varepsilon^{\mu\nu\rho\lambda}f_{m[ab}\frac{\partial W_{pq]}}{\partial\varphi_m}A^{b}_{\lambda}\eta^p\eta^q\Bigg) + \frac{g}{2}\frac{\partial W_{ab}}{\partial\varphi_c}\eta^{*}_{c}\eta^a\eta^b \\
& + gH^{*\mu}_{d}H^{*\nu}_{e}\left(\frac{\partial^2 W_{ab}}{\partial\varphi_d\partial\varphi_e}\left(\eta^a C^{b}_{\mu\nu} - 3A^{a\rho}C^{b}_{\mu\nu\rho}\right) \right.\\
& + \frac{\partial^3 W_{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e} \\
& \times\left(3A^{a\rho}A^{b\lambda}\eta_{c\mu\nu\rho\lambda} - \frac{1}{4}\eta^a\eta^b B_{c\mu\nu} + \frac{3}{2}A^{a\rho}\eta^b\eta_{c\mu\nu\rho}\right) \\
& \left. + \frac{3}{4\cdot 4!}\varepsilon_{\mu\nu\rho\lambda}f_{a[mn}\frac{\partial^3 W_{pq]}}{\partial\varphi_a\partial\varphi_d\partial\varphi_e}A^{m\rho}A^{n\lambda}\eta^p\eta^q\right) \\
& - 3gB^{*a}_{\mu\nu}B^{*b}_{\rho\lambda} \\
& \times\left(2\frac{\partial W_{ab}}{\partial\varphi_c}\eta^{\mu\nu\rho\lambda}_{c} + \frac{1}{2\cdot 4!}\varepsilon^{\mu\nu\rho\lambda}f_{c[ab}\frac{\partial W_{pq]}}{\partial\varphi_c}\eta^p\eta^q\right) \\
& - \frac{g}{2}\frac{\partial^2 W_{ab}}{\partial\varphi_c\partial\varphi_d}H^{*\mu}_{d}A^{*}_{c\mu}\eta^a\eta^b
\end{aligned}$$

$$
\begin{aligned}
&+ 3gH_{d}^{*\mu}B^{*a\nu\rho}\left(-2\frac{\partial W_{ab}}{\partial\varphi_d}C^{b}_{\mu\nu\rho}\right.\\
&+ \frac{\partial^2 W_{ab}}{\partial\varphi_c\partial\varphi_d}\left(\eta^b\eta_{c\mu\nu\rho}+4A^{b\lambda}\eta_{c\mu\nu\rho\lambda}\right)\\
&+ \left.\frac{1}{4!}\varepsilon_{\mu\nu\rho\lambda}f_{c[ab}\frac{\partial^2 W_{pq]}}{\partial\varphi_c\partial\varphi_d}A^{b\lambda}\eta^p\eta^q\right)\\
&+ C_{a}^{*\mu\nu\rho}\left(4\left(\hat{D}^{\lambda}\right)^{a}{}_{b}C^{b}_{\mu\nu\rho\lambda}-g\frac{\partial W_{bc}}{\partial\varphi_a}\eta^b C^{c}_{\mu\nu\rho}\right.\\
&+ g\frac{\partial^2 W_{bc}}{\partial\varphi_a\partial\varphi_d}\left(\frac{1}{4}\eta^b\eta^c\eta_{d\mu\nu\rho}-2A^{b\lambda}\eta^c\eta_{d\mu\nu\rho\lambda}\right)\\
&- \left.\frac{g}{3!\cdot 4!}\varepsilon_{\mu\nu\rho\lambda}f_{c[mn}\frac{\partial^2 W_{pq]}}{\partial\varphi_a\partial\varphi_c}A^{m\lambda}\eta^n\eta^p\eta^q\right)\\
&+ g\eta^{*a\mu\nu\rho\lambda}\left(2W_{ab}C^{b}_{\mu\nu\rho\lambda}\right.\\
&- \left.\frac{\partial W_{ab}}{\partial\varphi_c}\eta^b\eta_{c\mu\nu\rho\lambda}-\frac{2}{(4!)^2}\varepsilon_{\mu\nu\rho\lambda}f_{b[an}\frac{\partial W_{pq]}}{\partial\varphi_b}\eta^n\eta^p\eta^q\right)\\
&+ g\left(\frac{\partial^2 W_{ab}}{\partial\varphi_c\partial\varphi_d}H_{c}^{*[\mu}C_{d}^{*\nu\rho]}+\frac{\partial^3 W_{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e}H_{c}^{*\mu}H_{d}^{*\nu}H_{e}^{*\rho}\right)\\
&\times\left(-\eta^a C^{b}_{\mu\nu\rho}+4A^{a\lambda}C^{b}_{\mu\nu\rho\lambda}\right)\\
&+ 4g\left(3\left(\frac{\partial W_{ab}}{\partial\varphi_c}C_{c}^{*\mu\nu}+\frac{\partial^2 W_{ab}}{\partial\varphi_c\partial\varphi_d}H_{c}^{*\mu}H_{d}^{*\nu}\right)B^{*a\rho\lambda}\right.\\
&+ \left.2\frac{\partial W_{ab}}{\partial\varphi_c}H_{c}^{*\mu}\eta^{*a\nu\rho\lambda}\right)C^{b}_{\mu\nu\rho\lambda}\\
&+ \frac{1}{2}g\left(\frac{\partial^3 W_{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e}H_{d}^{*[\mu}C_{e}^{*\nu\rho]}\right.\\
&+ \left.\frac{\partial^4 W_{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e\partial\varphi_f}H_{d}^{*\mu}H_{e}^{*\nu}H_{f}^{*\rho}\right)\\
&\times\left(\frac{1}{2}\eta^a\eta^b\eta_{c\mu\nu\rho}-4A^{a\lambda}\eta^b\eta_{c\mu\nu\rho\lambda}\right)\\
&- 2g\left(3\left(\frac{\partial^2 W_{ab}}{\partial\varphi_c\partial\varphi_d}C_{d}^{*\mu\nu}+\frac{\partial^3 W_{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e}H_{d}^{*\mu}H_{e}^{*\nu}\right)B^{*a\rho\lambda}\right.\\
&+ \left.2\frac{\partial^2 W_{ab}}{\partial\varphi_c\partial\varphi_d}H_{d}^{*\mu}\eta^{*a\nu\rho\lambda}\right)\eta^b\eta_{c\mu\nu\rho\lambda}\\
&- \frac{g}{(4!)^2}\varepsilon_{\mu\nu\rho\lambda}f_{a[mn}\left(4\left(\frac{\partial^3 W_{pq]}}{\partial\varphi_a\partial\varphi_c\partial\varphi_d}H_{c}^{*[\mu}C_{d}^{*\nu\rho]}\right.\right.\\
&+ \left.\frac{\partial^4 W_{pq]}}{\partial\varphi_a\partial\varphi_c\partial\varphi_d\partial\varphi_e}H_{c}^{*\mu}H_{d}^{*\nu}H_{e}^{*\rho}\right)A^{m\lambda}\\
&+ 12\left(\frac{\partial^2 W_{pq]}}{\partial\varphi_a\partial\varphi_c}C_{c}^{*\mu\nu}+\frac{\partial^3 W_{pq]}}{\partial\varphi_a\partial\varphi_c\partial\varphi_d}H_{c}^{*\mu}H_{d}^{*\nu}\right)B^{*m\rho\lambda}\\
&+ \left.8\frac{\partial^2 W_{pq]}}{\partial\varphi_a\partial\varphi_c}H_{c}^{*\mu}\eta^{*m\nu\rho\lambda}\right)\eta^n\eta^p\eta^q\\
&+ gC_{c}^{*\mu\nu\rho\lambda}\left(\frac{\partial W_{ab}}{\partial\varphi_c}\eta^a C^{b}_{\mu\nu\rho\lambda}-\frac{1}{4}\frac{\partial^2 W_{ab}}{\partial\varphi_c\partial\varphi_d}\eta^a\eta^b\eta_{d\mu\nu\rho\lambda}\right.\\
&- \left.\frac{1}{4\cdot(4!)^2}\varepsilon_{\mu\nu\rho\lambda}f_{a[mn}\frac{\partial^2 W_{pq]}}{\partial\varphi_a\partial\varphi_c}\eta^m\eta^n\eta^p\eta^q\right)\\
&+ g\left(\frac{\partial^2 W_{ab}}{\partial\varphi_c\partial\varphi_d}\left(H_{c}^{*[\mu}C_{d}^{*\nu\rho\lambda]}+C_{c}^{*[\mu\nu}C_{d}^{*\rho\lambda]}\right)\right.\\
&+ \frac{\partial^3 W_{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e}H_{c}^{*[\mu}H_{d}^{*\nu}C_{e}^{*\rho\lambda]}\\
&+ \left.\frac{\partial^4 W_{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e\partial\varphi_f}H_{c}^{*\mu}H_{d}^{*\nu}H_{e}^{*\rho}H_{f}^{*\lambda}\right)\eta^a C^{b}_{\mu\nu\rho\lambda}\\
&- \frac{1}{4}g\left(\frac{\partial^3 W_{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e}\left(H_{d}^{*[\mu}C_{e}^{*\nu\rho\lambda]}+C_{d}^{*[\mu\nu}C_{e}^{*\rho\lambda]}\right)\right.\\
&+ \frac{\partial^4 W_{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e\partial\varphi_f}H_{d}^{*[\mu}H_{e}^{*\nu}C_{f}^{*\rho\lambda]}\\
&+ \left.\frac{\partial^5 W_{ab}}{\partial\varphi_c\partial\varphi_d\partial\varphi_e\partial\varphi_f\partial\varphi_g}H_{d}^{*\mu}H_{e}^{*\nu}H_{f}^{*\rho}H_{g}^{*\lambda}\right)\eta^a\eta^b\eta_{c\mu\nu\rho\lambda}\\
&- \frac{g}{4\cdot(4!)^2}\varepsilon_{\mu\nu\rho\lambda}\\
&\times f_{a[mn}\left(\frac{\partial^3 W_{pq]}}{\partial\varphi_a\partial\varphi_c\partial\varphi_d}\left(H_{c}^{*[\mu}C_{d}^{*\nu\rho\lambda]}+C_{c}^{*[\mu\nu}C_{d}^{*\rho\lambda]}\right)\right.\\
&+ \frac{\partial^4 W_{pq]}}{\partial\varphi_a\partial\varphi_c\partial\varphi_d\partial\varphi_e}H_{c}^{*[\mu}H_{d}^{*\nu}C_{e}^{*\rho\lambda]}\\
&+ \left.\left.\frac{\partial^5 W_{pq]}}{\partial\varphi_a\partial\varphi_c\partial\varphi_d\partial\varphi_e\partial\varphi_f}H_{c}^{*\mu}H_{d}^{*\nu}H_{e}^{*\rho}H_{f}^{*\lambda}\right)\eta^m\eta^n\eta^p\eta^q\right),
\end{aligned}
\tag{102}
$$

where we used the notation

$$\hat{D}^{\mu}\varphi_a=\partial^{\mu}\varphi_a+gW_{ab}A^{b\mu},\tag{103}$$

$$\hat{F}^{a}_{\mu\nu}=\partial_{[\mu}A^{a}_{\nu]}+g\frac{\partial W_{bc}}{\partial\varphi_a}A^{b}_{\mu}A^{c}_{\nu},\tag{104}$$

$$\left(\hat{D}_{\mu}\right)^{a}{}_{b}=\delta^{a}_{b}\partial_{\mu}-g\frac{\partial W_{bc}}{\partial\varphi_a}A^{c}_{\mu},\tag{105}$$

$$\left(\hat{D}_{\mu}\right)_{b}{}^{a}=\delta^{a}_{b}\partial_{\mu}+g\frac{\partial W_{bc}}{\partial\varphi_a}A^{c}_{\mu}.\tag{106}$$

From the full solution (102) we can extract all the information on the resulting interacting model.

Indeed, the pieces with antighost number zero from (102) produce the *Lagrangian action of the coupled theory*

$$
\begin{aligned}
\bar{S}_0^{(\mathrm{I.a})}=\int \mathrm{d}^4x\,&\left(H^{a}_{\mu}\hat{D}^{\mu}\varphi_a+\frac{1}{2}B^{\mu\nu}_{a}\hat{F}^{a}_{\mu\nu}\right.\\
&-\frac{g}{4\cdot 4!}\varepsilon^{\mu\nu\rho\lambda}f_{m[ab}\frac{\partial W_{cd]}}{\partial\varphi_m}A^{a}_{\mu}A^{b}_{\nu}A^{c}_{\rho}A^{d}_{\lambda}\\
&+\left.\mathcal{L}_0\left(\left[y^i\right]\right)+gj^{\mu}_{a}\left(y^i\right)k^{ae}W_{eb}\left(\varphi\right)A^{b}_{\mu}\right).
\end{aligned}
\tag{107}
$$

The first three types of terms describe the self-interactions of the BF fields. They were partially obtained by us in [12] under the supplementary assumption that the interactions do not break the PT invariance (which is the same with setting $f_{mab}=0$ in (107)). Here, we dropped this requirement and consequently allowed the appearance of a new vertex that is quartic in the one-forms A^{a}_{μ}. The last term provides the couplings between the BF fields and the matter fields and can be written in the compact form $g\bar{j}^{\mu}_{b}A^{b}_{\mu}$, where the bar current is given in (74), with the functions U^{a}_{b} as in (94). It is interesting to remark that we have a generalized minimal coupling, in the sense that even if it is formally expressed like "vector fields times currents", however the currents are not the conserved, purely matter ones from (69), but the matter currents in some "background" potentials $k^{ae}W_{eb}$ of scalar fields.

With the help of the elements of antighost number equal to one present in (102) we read the *generating set of deformed gauge transformations* for the action (107), which can be obtained by detaching the antifields from these terms and by replacing the ghosts with the corresponding gauge parameters

$$\hat{\delta}_\epsilon \varphi_a = -gW_{ab}\epsilon^b,\ \hat{\delta}_\epsilon A^a_\mu = \left(\hat{D}_\mu\right)^a_{\ b} \epsilon^b, \tag{108}$$

$$\begin{aligned}\hat{\delta}_\epsilon H^a_\mu &= 2\left(\hat{D}^\nu\right)^a_{\ b}\epsilon^b_{\mu\nu} + g\left(j_{m\mu}k^{me}\frac{\partial W_{eb}}{\partial\varphi_a} - \frac{\partial W_{bc}}{\partial\varphi_a}H^c_\mu\right.\\ &\quad + \frac{\partial^2 W_{bd}}{\partial\varphi_a\partial\varphi_c}A^{d\nu}B_{c\mu\nu}\\ &\quad \left.+ \frac{1}{4!}\varepsilon_{\mu\nu\rho\lambda}f_{c[mn}\frac{\partial^2 W_{pb]}}{\partial\varphi_a\partial\varphi_c}A^{m\nu}A^{n\rho}A^{p\lambda}\right)\epsilon^b\\ &\quad - \frac{3}{2}g\frac{\partial^2 W_{bd}}{\partial\varphi_a\partial\varphi_c}A^{b\nu}A^{d\rho}\epsilon_{c\mu\nu\rho}\end{aligned} \tag{109}$$

$$\begin{aligned}\hat{\delta}_\epsilon B^{\mu\nu}_a &= -3\left(\hat{D}_\rho\right)_a^{\ b}\epsilon_b^{\mu\nu\rho} + 2gW_{ab}\epsilon^{b\mu\nu}\\ &\quad - g\left(\frac{\partial W_{ab}}{\partial\varphi_c}B^{\mu\nu}_c - \frac{1}{8}\varepsilon^{\mu\nu\rho\lambda}f_{d[ac}\frac{\partial W_{pb]}}{\partial\varphi_d}A^c_\rho A^p_\lambda\right)\epsilon^b,\end{aligned} \tag{110}$$

$$\hat{\delta}_\epsilon y^i = g\left(T_a\right)^i_{\ j} y^j k^{ae}W_{eb}\epsilon^b. \tag{111}$$

All the gauge transformations are deformed with respect to the free ones, (2) and (3). The striking features of the new gauge symmetries can be synthesized by
(1) the matter fields gain gauge transformations, which can be viewed as being obtained by gauging the rigid symmetries (70) with the choice (79) and by further putting them in the same "background" potentials $k^{ae}W_{eb}$ of scalar fields;
(2) the BF one-forms H^a_μ are endowed with gauge transformations that are proportional with the matter currents and with the derivatives of the "background" potentials. Since the deformed gauge generators depend on the fields, in contrast to the "free" theory (1), we expect that the corresponding gauge algebra is non-Abelian, although the matrices T_a commute (the second relation in (94)). As it will be seen below, this is indeed the case. Another interesting observation is that the deformed field strengths and gauge transformations of the one-forms A^a_μ take generalized Yang–Mills forms, with the usual structure constants replaced by the derivatives of the "background" potentials (see (104) and the second relation in (108), with the corresponding "covariant derivatives" as in (105)).

The pieces of antighost number two are known to describe the *deformed gauge algebra and new first-order reducibility relations*. They belong to four distinct categories. Firstly, those linear in the antifields of the ghosts and quadratic in the ghosts with the pure ghost number equal to one contain the structure functions corresponding to the commutators among the gauge transformations (108)–(111). Secondly, the presence of elements which are simultaneously quadratic in the antifields of the original fields as well as in the ghosts with the pure ghost number equal to one shows that the gauge transformations (108)–(111) only close on-shell. Here "on-shell" means on the stationary surface of field equations for the deformed action (107). As a consequence, it follows that *the deformed gauge algebra for the type I solutions is open*, i.e., only closes on-shell, unlike the initial one, which is Abelian. It is interesting to mention that the matrices T_a from the gauge transformations (111) of the matter fields are commuting (see the second relation in (94)), but the matter gauge transformations are not. This is essentially so because (111) also involves the functions W_{eb} that satisfy the identities (91). Thirdly, there appear terms which are linear both in the antifields of the ghosts and in the ghosts with the pure ghost number equal to two; the functions that "glue" these BRST generators are precisely the deformed first-order reducibility functions of the coupled model. Fourthly, we notice the existence of pieces that are quadratic in the antifields of the original fields and also linear in the ghosts with the pure ghost number equal to two – they exhibit the on-shell closeness of the first-order reducibility relations, in contrast to the initial theory, whose reducibility takes place everywhere on the space of field histories. The terms with the antighost number equal to three and four complete the tensor gauge structure of the interacting model. Among others, they lead to the conclusion that the reducibility relations of order two also hold only on-shell.

I.b. In the opposite situation, where the conserved matter currents are not invariant under the gauge version of (70),

$$\frac{\delta^R j^\mu_a}{\delta y^i}\left(T_b\right)^i_{\ j} y^j \neq 0, \tag{112}$$

it follows that (S_1, S_1) is not vanishing; hence the second-order deformation S_2 as solution to (33) will also be so. This happens for instance if the matter currents contain spacetime derivatives. Moreover, it is possible to obtain other non-trivial, higher-order deformations when solving the remaining equations ((34) etc.). The expressions of these deformations strongly depend on the structure of the matter theory and cannot be output in the general setting considered here. What is always valid is that the complete deformed solution to the master equation starts like

$$\bar{S}^{(\mathrm{I.b})} = \bar{S}^{(\mathrm{I.a})} + g^2 S_2^{(\mathrm{I.b})} + \mathcal{O}\left(g^3\right). \tag{113}$$

Accordingly, the Lagrangian action of the coupled gauge theory will contain interactions of order g^2 and possibly of higher orders.

4.2.2 Type II solutions

Inserting (95)–(98) in (92) we deduce that (S_1, S_1) becomes

$$\begin{aligned}&(S_1, S_1)\\ &= \int \mathrm{d}^4x\left(2\left(j^\mu_c \bar{f}^c_{ab} + \frac{\delta^R j^\mu_a}{\delta y^i}\left(T_b\right)^i_{\ j} y^j\right)A^a_\mu\eta^b + \Lambda_{\mathrm{II}}\right),\end{aligned} \tag{114}$$

with $\Lambda_{\rm II}$ denoting the restriction of Λ to the type II solutions. Again, we consider two basic situations.
II.a. Assume that

$$j_c^\mu \bar{f}_{ab}^c + \frac{\delta^R j_a^\mu}{\delta y^i} \left(T_b\right)^i{}_j y^j = 0, \tag{115}$$

or, which is the same, that the conserved matter currents transform under the gauge version of the rigid symmetries (70) according to the adjoint representation of the Lie algebra $L(\mathcal{G})$. Similar to subcase I.a, the relation (115) might hold only if the currents j_a^μ have no derivatives. This is so since if we take $\bar{f}_{ab}^c = 0$ in (115), then we arrive precisely at (101). In consequence, in this case we have $\Lambda_{\rm II} = 0$, which further leads to $(S_1, S_1) = 0$, so we can set $S_2 = 0$. Moreover, all the higher-order equations (34) etc. are satisfied with the choice $S_3 = S_4 = \cdots = 0$. Consequently, the deformed solution to the master equation *consistent to all orders in the coupling constant* is provided by (78) where we use (79), (90) and (95)–(98), and reduces to

$$\begin{aligned}
\overline{S}^{(\mathrm{II.a})} = &\int \mathrm{d}^4x \left(H_\mu^a D^\mu \varphi_a + \mathcal{L}_0\left(\left[y^i\right]\right) + g j_a^\mu A_\mu^a\right.\\
&- \frac{g}{4\cdot 4!}\varepsilon^{\mu\nu\rho\lambda} f_{e[ab}\bar{f}_{cd]}^e A_\mu^a A_\nu^b A_\rho^c A_\lambda^d + \frac{1}{2}B_a^{\mu\nu}\bar{F}_{\mu\nu}^a\\
&- g\varphi^{*a}\left(\bar{f}_{ab}^c\varphi_c + F_{ab}\right)\eta^b + g y_i^*\left(T_a\right)^i{}_j y^j \eta^a\\
&+ A_a^{*\mu}\left(D_\mu\right)^a{}_b \eta^b + H_a^{*\mu}\left(2\left(D^\nu\right)^a{}_b C_{\mu\nu}^b - g\bar{f}_{bc}^a\eta^b H_\mu^c\right)\\
&+ \frac{g}{2}\eta_c^*\bar{f}_{ab}^c\eta^a\eta^b\\
&+ B_{\mu\nu}^{*a}\left(-3\left(D_\rho\right)_a{}^b \eta_b^{\mu\nu\rho} - g\bar{f}_{ab}^c\eta^b B_c^{\mu\nu}\right.\\
&+ 2g\left(\bar{f}_{ab}^c\varphi_c + F_{ab}\right)C^{b\mu\nu}\\
&\left.+ \frac{g}{8}\varepsilon^{\mu\nu\rho\lambda} f_{e[ab}\bar{f}_{cd]}^e A_\rho^b A_\lambda^c \eta^d\right)\\
&- 6g\bar{f}_{ab}^c\left(H_c^{*\mu}B^{*a\nu\rho}C_{\mu\nu\rho}^b + B_{\mu\nu}^{*a}B_{\rho\lambda}^{*b}\eta_c^{\mu\nu\rho\lambda}\right)\\
&- \frac{3g}{2\cdot 4!}\varepsilon^{\mu\nu\rho\lambda} f_{e[ab}\bar{f}_{cd]}^e B_{\mu\nu}^{*a}B_{\rho\lambda}^{*b}\eta^c\eta^d\\
&+ C_a^{*\mu\nu}\left(-3\left(D^\rho\right)^a{}_b C_{\mu\nu\rho}^b + g\bar{f}_{bc}^a\eta^b C_{\mu\nu}^c\right)\\
&+ \eta_{\mu\nu\rho}^{*a}\left(4\left(D_\lambda\right)_a{}^b \eta_b^{\mu\nu\rho\lambda} + g\bar{f}_{ab}^c\eta^b\eta_c^{\mu\nu\rho}\right.\\
&\left.- 2g\left(\bar{f}_{ab}^c\varphi_c + F_{ab}\right)C^{b\mu\nu\rho} + \frac{g}{4!}\varepsilon^{\mu\nu\rho\lambda} f_{e[ab}\bar{f}_{cd]}^e A_\lambda^b\eta^c\eta^d\right)\\
&+ C_a^{*\mu\nu\rho}\left(4\left(D^\lambda\right)^a{}_b C_{\mu\nu\rho\lambda}^b - g\bar{f}_{bc}^a\eta^b C_{\mu\nu\rho}^c\right)\\
&+ g\eta_{\mu\nu\rho\lambda}^{*a}\left(-\bar{f}_{ab}^c\eta^b\eta_c^{\mu\nu\rho\lambda} + 2\left(\bar{f}_{ab}^c\varphi_c + F_{ab}\right)C^{b\mu\nu\rho\lambda}\right.\\
&\left.- \frac{2}{(4!)^2}\varepsilon^{\mu\nu\rho\lambda} f_{e[ab}\bar{f}_{cd]}^e \eta^b\eta^c\eta^d\right)\\
&+ g\bar{f}_{ab}^c\left(4\left(3C_c^{*\mu\nu}B^{*a\rho\lambda} + 2H_c^{*\mu}\eta^{*a\nu\rho\lambda}\right)\right.\\
&\left.\left.+ C_c^{*\mu\nu\rho\lambda}\eta^a\right)C_{\mu\nu\rho\lambda}^b\right),
\end{aligned} \tag{116}$$

where we used the notation

$$D^\mu\varphi_a = \partial^\mu\varphi_a + g\left(\bar{f}_{ab}^c\varphi_c + F_{ab}\right)A^{b\mu}, \tag{117}$$

$$\overline{F}_{\mu\nu}^a = \partial_{[\mu}A_{\nu]}^a + g\bar{f}_{bc}^a A_\mu^b A_\nu^c, \tag{118}$$

$$\left(D_\mu\right)^a{}_b = \delta_b^a\partial_\mu - g\bar{f}_{bc}^a A_\mu^c, \tag{119}$$

$$\left(D_\mu\right)_a{}^b = \delta_a^b\partial_\mu + g\bar{f}_{ac}^b A_\mu^c. \tag{120}$$

The Lagrangian formulation of the interacting model is deduced from (116) following the same pattern as in the previous subsubsection, i.e. analyzing its components with definite, increasing antighost numbers.

Thus, the *Lagrangian action of the interacting theory* takes the simpler form

$$\begin{aligned}
\bar{S}_0^{(\mathrm{II.a})} = &\int \mathrm{d}^4x \left(H_\mu^a D^\mu\varphi_a + \frac{1}{2}B_a^{\mu\nu}\overline{F}_{\mu\nu}^a\right.\\
&- \frac{g}{4\cdot 4!}\varepsilon^{\mu\nu\rho\lambda} f_{e[ab}\bar{f}_{cd]}^e A_\mu^a A_\nu^b A_\rho^c A_\lambda^d\\
&\left.+ \mathcal{L}_0\left(\left[y^i\right]\right) + g j_a^\mu\left(y^i\right)A_\mu^a\right),
\end{aligned} \tag{121}$$

and it is invariant under the *deformed (generating set of) gauge transformations*

$$\bar{\delta}_\epsilon\varphi_a = -g\left(\bar{f}_{ab}^c\varphi_c + F_{ab}\right)\epsilon^b,$$

$$\bar{\delta}_\epsilon A_\mu^a = \left(D_\mu\right)^a{}_b\epsilon^b, \tag{122}$$

$$\bar{\delta}_\epsilon H_\mu^a = 2\left(D^\nu\right)^a{}_b\epsilon_{\mu\nu}^b - g\bar{f}_{bc}^a\epsilon^b H_\mu^c, \tag{123}$$

$$\begin{aligned}
\bar{\delta}_\epsilon B_a^{\mu\nu} = &-3\left(D_\rho\right)_a{}^b\epsilon_b^{\mu\nu\rho} - g\bar{f}_{ab}^c\epsilon^b B_c^{\mu\nu}\\
&+ 2g\left(\bar{f}_{ab}^c\varphi_c + F_{ab}\right)\epsilon^{b\mu\nu}\\
&+ \frac{g}{8}\varepsilon^{\mu\nu\rho\lambda} f_{e[ab}\bar{f}_{cd]}^e A_\rho^b A_\lambda^c\epsilon^d,
\end{aligned} \tag{124}$$

$$\bar{\delta}_\epsilon y^i = g\left(T_a\right)^i{}_j y^j\epsilon^a. \tag{125}$$

Let us briefly comment on the physical features of the coupled model in this situation. The first three terms from (121) describe again the self-interactions among the BF fields. The abelian field strengths of the one-forms $\left\{A_\mu^a\right\}$ are now deformed into the standard Yang–Mills form (118), instead of the more general expression (104) from the previous case. Also, the coupling between the BF and the matter sector is a minimal one, of the form "vector fields times currents", where the currents are now precisely the conserved, purely matter ones from (69). Along the same lines, we read that the new gauge transformations of the one-forms $\left\{A_\mu^a\right\}$ are standard Yang–Mills, being given by the second relation in (122), with the corresponding covariant derivative expressed by (119). The matter fields are endowed, as a consequence of the type II solutions, with the deformed gauge transformations (125), which are nothing but the gauge version of the rigid ones (70) with the generators as in (79) and satisfying the relation (98). We would expect that the deformed gauge algebra be precisely the Lie algebra $L(\mathcal{G})$ with the structure constants $-\bar{f}_{ab}^c$. However, this is not true since (116) contains a term proportional to $\varepsilon^{\mu\nu\rho\lambda} f_{e[ab}\bar{f}_{cd]}^e B_{\mu\nu}^{*a}B_{\rho\lambda}^{*b}\eta^c\eta^d$, which shows that the *deformed gauge algebra is in fact open*, so it closes only on-shell. (More precisely, it closes when the field equations for the two-forms hold, $\delta\bar{S}_0^{(\mathrm{II.a})}/\delta B_a^{\mu\nu} \approx 0$.) This is a relatively rare example of gauge theory with an open gauge algebra, whose structure functions reduce entirely to the structure constants of a Lie algebra. If we however forbid the breaking of PT invariance, namely take $f_{eab} = 0$ in (116), then the deformed gauge algebra indeed becomes the Lie algebra $L(\mathcal{G})$. As for the redundancy of the new gauge transformations, it remains of order two, with *the corresponding deformed reducibility relations closing on-shell*.

II.b. In the opposite situation, where the conserved matter currents do not transform under the gauge version of the rigid symmetries (70) according to the adjoint representation of the Lie algebra $L(\mathcal{G})$:

$$j^{\mu}_{c}\bar{f}^{c}_{ab} + \frac{\delta^{R} j^{\mu}_{a}}{\delta y^{i}} (T_b)^{i}{}_{j}\, y^{j} \neq 0, \qquad (126)$$

we find ourselves in the same framework as in subcase I.b. Thus, (S_1, S_1) is not vanishing, such that the second-order deformation S_2 involved in (33) will also be so. In principle, it is possible to infer other non-trivial higher-order deformations as solutions to the equations ((34) etc.). The concrete form of these deformations depends again on the structure of the matter theory and cannot be prescribed here. We can only write that the complete deformed solution to the master equation begins like

$$\bar{S}^{(\mathrm{II.b})} = \bar{S}^{(\mathrm{II.a})} + g^{2} S_{2}^{(\mathrm{II.b})} + \mathcal{O}\left(g^{3}\right), \qquad (127)$$

such that the interacting Lagrangian action includes vertices of order g^2, and possibly of higher orders.

Finally, a word of caution. Once the deformations related to a given matter theory are computed, special attention should be paid to the elimination of non-locality, as well as of triviality of the resulting deformations. This completes our general deformation procedure, based on local BRST cohomology.

5 Examples

Next, we consider two examples of matter theories – Dirac fields and real scalar fields – and determine their consistent interactions with the four-dimensional BF model under discussion in the light of the analysis performed in the previous sections.

5.1 Couplings for a set of Dirac fields

First, we examine the consistent couplings with a collection of massive Dirac fields. In view of this, we start from the Lagrangian action of the matter fields $S_0^{\mathrm{matt}}\left[y^i\right]$ in (1) of the form

$$S_0^{\mathrm{matt}}\left[\psi^{\alpha}_{A}, \bar{\psi}^{A}_{\alpha}\right] = \int \mathrm{d}^4x \left(\bar{\psi}^{A}_{\alpha}\left(\mathrm{i}\left(\gamma^{\mu}\right)^{\alpha}{}_{\beta}\,\partial_{\mu} - m\delta^{\alpha}_{\beta}\right)\psi^{\beta}_{A}\right), \qquad (128)$$

where ψ^{α}_{A} and $\bar{\psi}^{A}_{\alpha}$ ($A = \overline{1, I'}$, $\alpha = 1,2,3,4$) denote the spinor components of the complex Dirac spinors ψ_A and $\bar{\psi}^A$ (the bar operation signifies spinor conjugation). The actions of δ and γ on the matter generators from the free BRST complex are expressed by

$$\delta\psi^{\alpha}_{A} = 0,\ \delta\bar{\psi}^{A}_{\alpha} = 0, \qquad (129)$$

$$\delta\psi^{*A}_{\alpha} = -\left(\mathrm{i}\left(\gamma^{\mu}\right)^{\beta}{}_{\alpha}\,\partial_{\mu} + m\delta^{\beta}_{\alpha}\right)\bar{\psi}^{A}_{\beta}, \qquad (130)$$

$$\delta\bar{\psi}^{*\alpha}_{A} = -\left(\mathrm{i}\left(\gamma^{\mu}\right)^{\alpha}{}_{\beta}\,\partial_{\mu} - m\delta^{\alpha}_{\beta}\right)\psi^{\beta}_{A}, \qquad (131)$$

$$\gamma\psi^{\alpha}_{A} = 0,\ \gamma\bar{\psi}^{A}_{\alpha} = 0,\ \gamma\psi^{*A}_{\alpha} = 0,\ \gamma\bar{\psi}^{*\alpha}_{A} = 0, \qquad (132)$$

where the antighost number one antifields ψ^{*A}_{α} and $\bar{\psi}^{*\alpha}_{A}$ are bosonic. Their actions on the variables from the BF sector are correctly defined by the appropriate relations in (18)–(26). Let us consider the rigid symmetries

$$\Delta_{\xi}\bar{\psi}^{A}_{\alpha} = \bar{\psi}^{B}_{\alpha}\left(T_a\right)^{A}{}_{B}\,\xi^{a},\ \Delta_{\xi}\psi^{\alpha}_{A} = -\left(T_a\right)^{B}{}_{A}\,\psi^{\alpha}_{B}\xi^{a}, \qquad (133)$$

of the action (128), such that the corresponding conserved currents read

$$j^{\mu}_{a} = \mathrm{i}\bar{\psi}^{A}_{\alpha}\left(\gamma^{\mu}\right)^{\alpha}{}_{\beta}\left(T_a\right)^{B}{}_{A}\,\psi^{\beta}_{B}. \qquad (134)$$

From (134) we find that

$$\frac{\delta^{R} j^{\mu}_{a}}{\delta y^{i}}\left(T_b\right)^{i}{}_{j}\,y^{j} = \frac{\delta^{R} j^{\mu}_{a}}{\delta\bar{\psi}^{A}_{\alpha}}\left(T_b\right)^{A}{}_{B}\,\bar{\psi}^{B}_{\alpha} - \frac{\delta^{R} j^{\mu}_{a}}{\delta\psi^{\alpha}_{A}}\left(T_b\right)^{B}{}_{A}\,\psi^{\alpha}{}_{B}, \qquad (135)$$

and hence

$$\frac{\delta^{R} j^{\mu}_{a}}{\delta\bar{\psi}^{A}_{\alpha}}\left(T_b\right)^{A}{}_{B}\,\bar{\psi}^{B}_{\alpha} - \frac{\delta^{R} j^{\mu}_{a}}{\delta\psi^{\alpha}_{A}}\left(T_b\right)^{B}{}_{A}\,\psi^{\alpha}{}_{B} = \mathrm{i}\bar{\psi}^{A}_{\alpha}\left(\gamma^{\mu}\right)^{\alpha}{}_{\beta}\left[T_a, T_b\right]^{B}{}_{A}\,\psi^{\beta}_{B}. \qquad (136)$$

In the case of type I solutions (see (94)), the relation (136) becomes

$$\frac{\delta^{R} j^{\mu}_{a}}{\delta\bar{\psi}^{A}_{\alpha}}\left(T_b\right)^{A}{}_{B}\,\bar{\psi}^{B}_{\alpha} - \frac{\delta^{R} j^{\mu}_{a}}{\delta\psi^{\alpha}_{A}}\left(T_b\right)^{B}{}_{A}\,\psi^{\alpha}{}_{B} = 0, \qquad (137)$$

so that we are under the conditions of subcase I.a. Consequently, (75) takes now the form

$$\bar{a}_1 = \left(\bar{\psi}^{A}_{\alpha}\bar{\psi}^{*\alpha}_{B} - \psi^{*A}_{\alpha}\psi^{\alpha}_{B}\right)\left(T_a\right)^{B}{}_{A}\,k^{ac}W_{cb}\eta^{b} + \mathrm{i}H^{*\mu}_{m}k^{ac}\frac{\partial W_{cb}}{\partial\varphi_m}\bar{\psi}^{A}_{\alpha}\left(\gamma_{\mu}\right)^{\alpha}{}_{\beta}\left(T_a\right)^{B}{}_{A}\,\psi^{\beta}_{B}\eta^{b}, \qquad (138)$$

while $\bar{a}_0$ (the solution to (63)) can be written as

$$\bar{a}_0 = \mathrm{i}\bar{\psi}^{A}_{\alpha}\left(\gamma^{\mu}\right)^{\alpha}{}_{\beta}\left(T_a\right)^{B}{}_{A}\,\psi^{\beta}_{B}k^{ac}W_{cb}A^{b}_{\mu}. \qquad (139)$$

According to the general theory, the consistency of the first-order deformation leads to no higher-order deformations in this situation. The deformed Lagrangian action and accompanying gauge transformations are given by (107) and respectively (108)–(111) where we set $y^i \rightarrow \psi^{\alpha}_{A}, \bar{\psi}^{A}_{\alpha}$ and use (133) and (134).

For the type II solutions (see (95)–(98)) it follows that the relation (136) leads to

$$\frac{\delta^{R} j^{\mu}_{a}}{\delta\bar{\psi}^{A}_{\alpha}}\left(T_b\right)^{A}{}_{B}\,\bar{\psi}^{B}_{\alpha} - \frac{\delta^{R} j^{\mu}_{a}}{\delta\psi^{\alpha}_{A}}\left(T_b\right)^{B}{}_{A}\,\psi^{\alpha}{}_{B} = -j^{\mu}_{c}\bar{f}^{c}_{ab}, \qquad (140)$$

so we are in subcase II.a. Furthermore, we have

$$\bar{a}_1 = \left(\bar{\psi}^{A}_{\alpha}\bar{\psi}^{*\alpha}_{B} - \psi^{*A}_{\alpha}\psi^{\alpha}_{B}\right)\left(T_a\right)^{B}{}_{A}\,\eta^{a}, \qquad (141)$$

$$\bar{a}_0 = \mathrm{i}\bar{\psi}^A_\alpha \left(\gamma^\mu\right)^\alpha_{\ \beta} \left(T_a\right)^B_{\ A} \psi^\beta_B A^a_\mu. \tag{142}$$

The consistency of the first-order deformation again produces no higher-order deformations. Similarly, the deformed action and its gauge transformations follow from the relations (121) and (122)–(125) where we replace y^i by ψ^α_A and $\bar{\psi}^A_\alpha$, and take into account the relations (133) and (134).

5.2 Couplings for a collection of real scalar fields

Second, we analyze the case where the role of the matter fields is played by a collection of real scalar fields, $\left\{\phi^A\right\}_{A=\overline{1,I}}$. In this situation the Lagrangian action of the matter fields from (1) is

$$S_0^{\mathrm{matt}}\left[\phi^A\right] = \int \mathrm{d}^4x \left(\frac{1}{2}K_{AB}\left(\partial_\mu\phi^A\right)\left(\partial^\mu\phi^B\right) - V\left(\phi^A\right)\right), \tag{143}$$

where K_{AB} is an invertible, symmetric, constant matrix. We assume that the matter action (143) is invariant under the bosonic rigid symmetries

$$\Delta_\xi\phi^A = -\left(T_a\right)^A_{\ B}\phi^B\xi^a. \tag{144}$$

This is true if the constant matrices T_a, of elements $\left(T_a\right)^A_{\ B}$, are such that the following relations are satisfied:

$$\frac{\partial V}{\partial\phi^A}\left(T_a\right)^A_{\ B}\phi^B = 0, \tag{145}$$

$$\left(\tilde{T}_a\right)_{AB} = -\left(\tilde{T}_a\right)_{BA}, \tag{146}$$

where

$$\left(\tilde{T}_a\right)_{AB} = K_{AE}\left(T_a\right)^E_{\ B}. \tag{147}$$

Assuming that such matrices exist, it follows that the conserved currents associated with the rigid symmetries (144) read

$$j^\mu_a = \left(\partial^\mu\phi^A\right)\phi^B\left(\tilde{T}_a\right)_{AB}. \tag{148}$$

For the model under consideration we then find that

$$\frac{\delta^R j^\mu_a}{\delta y^i}\left(T_b\right)^i_{\ j} y^j = -\frac{\delta^R j^\mu_a}{\delta\phi^A}\left(T_b\right)^A_{\ B}\phi^B. \tag{149}$$

With the help of the expression (148) we deduce that

$$\begin{aligned}
&-\frac{\delta^R j^\mu_a}{\delta\phi^A}\left(T_b\right)^A_{\ B}\phi^B \\
&= -\left[T_a,T_b\right]^C_{\ B}K_{AC}\left(\partial^\mu\phi^A\right)\phi^B \\
&\quad -\partial^\mu\left(\frac{1}{2}\left\{T_a,T_b\right\}^C_{\ B}K_{AC}\phi^A\phi^B\right),
\end{aligned} \tag{150}$$

where $\left\{T_a,T_b\right\} = T_aT_b + T_bT_a$.

For the type I solutions (see (94)) the relation (150) becomes

$$-\frac{\delta^R j^\mu_a}{\delta\phi^A}\left(T_b\right)^A_{\ B}\phi^B = -\partial^\mu\left(\frac{1}{2}\left\{T_a,T_b\right\}^C_{\ B}K_{AC}\phi^A\phi^B\right) \neq 0, \tag{151}$$

so we are in subcase I.b. In this context we have

$$\begin{aligned}
\bar{a}_1 = k^{ac}\Big(&-\phi^*_A\left(T_a\right)^A_{\ B}\phi^B W_{cb} \\
&+ H^{*\mu}_m\frac{\partial W_{cb}}{\partial\varphi_m}\left(\partial_\mu\phi^A\right)\left(\tilde{T}_a\right)_{AB}\phi^B\Big)\eta^b
\end{aligned} \tag{152}$$

and respectively

$$\bar{a}_0 = k^{ac}\left(\partial^\mu\phi^A\right)\left(\tilde{T}_a\right)_{AB}\phi^B W_{cb}A^b_\mu. \tag{153}$$

The consistency of the first-order deformation of the solution to the master equation leads to a non-trivial deformation at the second order, of the form

$$\begin{aligned}
S_2^{(\mathrm{I.b})} = &-\frac{1}{4}k^{ap}k^{de}K_{AC}\left\{T_a,T_d\right\}^C_{\ B}\int\mathrm{d}^4x\,\phi^A\phi^B \\
&\times\left(W_{pb}A^{b\mu} + H^{*\mu}_m\eta^b\frac{\partial W_{pb}}{\partial\varphi_m}\right)\left(W_{ec}A^c_\mu + H^*_{n\mu}\eta^c\frac{\partial W_{ec}}{\partial\varphi_n}\right).
\end{aligned} \tag{154}$$

Because of (102) adapted to our model and of (154) we obtain that $S_3^{(\mathrm{I.b})} = 0$ and also $S_4^{(\mathrm{I.b})} = S_5^{(\mathrm{I.b})} = \cdots = 0$. In this situation we get that the full deformed Lagrangian action is a polynomial of order two in the coupling constant

$$\begin{aligned}
\bar{S}_0^{(\mathrm{I.b})} = &\ \bar{S}_0^{(\mathrm{BFI})} \\
&+\int\mathrm{d}^4x\left[\frac{1}{2}K_{AB}\left(\left(\hat{D}_\mu\right)^A_{\ C}\phi^C\right)\left(\left(\hat{D}^\mu\right)^B_{\ D}\phi^D\right)\right. \\
&\left.\quad - V\left(\phi^A\right)\right],
\end{aligned} \tag{155}$$

where

$$\left(\hat{D}_\mu\right)^A_{\ C} = \delta^A_C\partial_\mu + g\left(T_a\right)^A_{\ C}k^{ab}W_{bc}A^c_\mu, \tag{156}$$

and $\bar{S}_0^{(\mathrm{BFI})}$ denotes the action that describes the self-interactions among the BF fields for the type I solutions, and reduces to the first three terms from the right-hand side of (107). The deformed gauge transformations of (155) are as in (108) and (110)–(111) where we set $y^i = \phi^A$ and $\left(T_a\right)^i_{\ j} = -\left(T_a\right)^A_{\ B}$, while the gauge transformations of the one-forms $\left\{H^a_\mu\right\}$ are enriched with terms of order two in the coupling constant:

$$\begin{aligned}
\hat{\delta}'_\epsilon H^a_\mu = &\ \hat{\delta}_\epsilon H^a_\mu \\
&+ g^2K_{AB}\left(T_b\right)^A_{\ C}\phi^C\left(T_e\right)^B_{\ D}\phi^D k^{ef}W_{fg}A^g_\mu k^{bc}\frac{\partial W_{cd}}{\partial\varphi_a}\epsilon^d.
\end{aligned} \tag{157}$$

In the above $\hat{\delta}_\epsilon H^a_\mu$ can be found in (109) and j^μ_a must be taken as in (148). The commutators among the deformed gauge transformations are also modified with terms of order two in the coupling constant, but the reducibility relations stop at order one in the coupling constant and hence take the same form as in subcase I.a.

For the type II solutions (see (95)–(98)) the relation (150) gives

$$-\frac{\delta^R j_a^\mu}{\delta \phi^A} (T_b)^A_{\ B} \phi^B + \bar{f}^c_{ab} j^\mu_c$$
$$= -\partial^\mu \left(\frac{1}{2} \{T_a, T_b\}^C_{\ B} K_{AC} \phi^A \phi^B \right), \quad (158)$$

so we are in subcase II.b. Consequently, we have that

$$\bar{a}_1 = -\phi^*_A (T_a)^A_{\ B} \phi^B \eta^b \quad (159)$$

and respectively

$$\bar{a}_0 = \left(\partial^\mu \phi^A\right) \left(\tilde{T}_a\right)_{AB} \phi^B A^a_\mu. \quad (160)$$

The consistency of the first-order deformation leads to the second-order deformation

$$S_2^{(\mathrm{II.b})} = -\frac{1}{4} K_{AC} \{T_a, T_d\}^C_{\ B} \int \mathrm{d}^4 x \phi^A \phi^B A^{a\mu} A^d_\mu. \quad (161)$$

Using (116) adapted to the present model and (161) we get that $S_3^{(\mathrm{II.b})} = 0$ and, furthermore, $S_4^{(\mathrm{II.b})} = S_5^{(\mathrm{II.b})} = \cdots = 0$. Consequently, the complete Lagrangian action of the coupled model becomes

$$\bar{S}_0^{(\mathrm{II.b})} = \bar{S}_0^{(\mathrm{BFII})} \quad (162)$$
$$+ \int \mathrm{d}^4 x \left[\frac{1}{2} K_{AB} \left(\left(\bar{D}_\mu\right)^A_{\ C} \phi^C \right) \left(\left(\bar{D}^\mu\right)^B_{\ D} \phi^D \right) - V\left(\phi^A\right) \right],$$

where

$$\left(\bar{D}_\mu\right)^A_{\ C} = \delta^A_C \partial_\mu + g (T_a)^A_{\ C} A^a_\mu, \quad (163)$$

and $\bar{S}_0^{(\mathrm{BFII})}$ means the action responsible for the self-interactions among the BF fields for the type II solutions, being represented by the first three terms from the right-hand side of (121). It is again a polynomial of order two in the deformation parameter. In this situation the gauge transformations of the action (162) gain no new components of order two or higher in the coupling constant and are expressed as in (122)–(125) where we set $y^i = \phi^A$ and $(T_a)^i_{\ j} = -(T_a)^A_{\ B}$. Consequently, the gauge algebra and the reducibility relations are the same as in subcase II.a.

6 Conclusion

The main result of this paper is that we can indeed add consistent Lagrangian interactions to a "free" theory describing a collection of BF-like models and a matter theory in four dimensions. Our treatment is based on the deformation of the solution to the master equation. The first-order deformation is computed by means of the local BRST cohomology in ghost number zero. Its existence is due to the hypothesis that the matter theory is invariant under some (non-trivial) bosonic global symmetries, which produce some (non-trivially) conserved currents j^μ_a. The consistency of the first-order deformation restricts the commutators of the constant matrices T_a that enter the global matter symmetries to either vanish (type I solutions) or close according to a Lie algebra (type II solutions). The deformation procedure stops at order one if the matter currents j^μ_a include no derivatives and if they either remain invariant under the gauge version of the rigid symmetries in the first case or transform under the gauge version according to the adjoint representation of $L(\mathcal{G})$ in the second case. Otherwise, there appear deformations of order g^2 and possibly of higher orders.

The common features of the interacting gauge models resulting from the two types of solutions at order g are as follows: the matter fields are primarily coupled to the vector fields A^a_μ; all the fields (BF and matter) gain deformed gauge transformations; the gauge algebra of the deformed gauge transformations closes on-shell (in spite of the Abelian and respectively Lie character of the matrices T_a), in contrast to the "free", Abelian one; the reducibility relations hold on-shell, i.e. on the stationary surface of deformed field equations, unlike the initial ones, that held off-shell. The main differences between the two cases are revealed by the couplings of the matter fields to the BF sector and by the expressions of the gauge transformations. Indeed, for the type I solutions we find a generalized minimal coupling in the sense that even if it is formally expressed like "vector fields A^a_μ times currents", however the currents are not the conserved matter currents j^μ_a, but j^μ_a in some "background" potentials of scalar fields proportional with W_{ab}, while for type II solutions we recover a genuine minimal coupling. The same observation holds for the gauge transformations of the matter fields: in the former case they can be viewed as being obtained by gauging the rigid matter symmetries and by further putting them in the same "background" potentials of scalar fields, while in the latter they reduce to gauging the rigid symmetries only. The deformed field strengths and gauge transformations of the one-forms A^a_μ inherit a similar behavior: they are generalized Yang–Mills-like for type I solutions, with the usual structure constants replaced by the derivatives of the "background" potentials, and standard Yang–Mills corresponding to the Lie algebra $L(\mathcal{G})$ for type II solutions. Finally, we note that the exemplification of our results in the case of a set of Dirac fields leads to no deformations of order two or higher for either type I or II solutions, while for a system of real scalar fields we obtain second-order deformations for both solutions.

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Appendix A: Notations used in Sect. 4.2

The various notations used within the formula (81) are listed below:

$$K^{abc} = \eta^a \eta^b \varphi^{*c} + 2\eta^a A^{b\mu} H^c_\mu$$
$$+ 2\left(A^{a\mu} A^{b\nu} - 2B^{*a\mu\nu} \eta^b\right) C^c_{\mu\nu}$$
$$+ 4\left(\eta^a \eta^{*b\mu\nu\rho} + 3B^{*a\mu\nu} A^{b\rho}\right) C^c_{\mu\nu\rho} \qquad \text{(A.1)}$$
$$- 4\left(\eta^a \eta^{*b\mu\nu\rho\lambda} + 6B^{*a\mu\nu} B^{*b\rho\lambda} - 4\eta^{*a\mu\nu\rho} A^{b\lambda}\right) C^c_{\mu\nu\rho\lambda},$$

$$K^{abc}_d = \left(4H^{*\nu}_d A^{a\mu} \eta^b - C^{*\mu\nu}_d \eta^a \eta^b\right) C^c_{\mu\nu} - H^{*\mu}_d \eta^a \eta^b H^c_\mu$$
$$+ \left(6H^{*\rho}_d A^{a\mu} A^{b\nu} - 12H^{*\rho}_d B^{*a\mu\nu} \eta^b + 6C^{*\mu\nu}_d \eta^a A^{b\rho}\right.$$
$$\left. - C^{*\mu\nu\rho}_d \eta^a \eta^b\right) C^c_{\mu\nu\rho} + \left(-48H^{*\lambda}_d B^{*a\mu\nu} A^{b\rho}\right.$$
$$+ 12C^{*\mu\nu}_d A^{a\rho} A^{b\lambda} + 16H^{*\lambda}_d \eta^{*a\mu\nu\rho} \eta^b - 24C^{*\mu\nu}_d B^{*a\rho\lambda} \eta^b$$
$$\left. - 8C^{*\mu\nu\rho}_d A^{a\lambda} \eta^b - C^{*\mu\nu\rho\lambda}_d \eta^a \eta^b\right) C^c_{\mu\nu\rho\lambda}, \qquad \text{(A.2)}$$

$$K^{abc}_{de} = -3\left(C^{*\mu\nu}_d H^{*\rho}_e \eta^a + 2H^{*\mu}_d H^{*\nu}_e A^{a\rho}\right) \eta^b C^c_{\mu\nu\rho}$$
$$- H^{*\mu}_d H^{*\nu}_e \eta^a \eta^b C^c_{\mu\nu} + \left(-24H^{*\mu}_d H^{*\nu}_e B^{*a\rho\lambda} \eta^b\right.$$
$$+ 12H^{*\mu}_d H^{*\nu}_e A^{a\rho} A^{b\lambda} - 24C^{*\mu\nu}_d H^{*\rho}_e A^{a\lambda} \eta^b$$
$$\left. - 3C^{*\mu\nu}_d C^{*\rho\lambda}_e \eta^a \eta^b + 4C^{*\mu\nu\rho}_d H^{*\lambda}_e \eta^a \eta^b\right) C^c_{\mu\nu\rho\lambda}, \qquad \text{(A.3)}$$

$$K^{abc}_{def} = -2\left(4H^{*\mu}_d H^{*\nu}_e H^{*\rho}_f A^{a\lambda}\right.$$
$$\left. + 3C^{*\mu\nu}_d H^{*\rho}_e H^{*\lambda}_f \eta^a\right) \eta^b C^c_{\mu\nu\rho\lambda}$$
$$- H^{*\mu}_d H^{*\nu}_e H^{*\rho}_f \eta^a \eta^b C^c_{\mu\nu\rho}, \qquad \text{(A.4)}$$

$$K^{abc}_{defg} = -H^{*\mu}_d H^{*\nu}_e H^{*\rho}_f H^{*\lambda}_{\mathrm{g}} \eta^a \eta^b C^c_{\mu\nu\rho\lambda}, \qquad \text{(A.5)}$$

$$U^{abc}_d = \left(-2\eta^a A^b_\mu A^c_\nu + B^{*a}_{\mu\nu} \eta^b \eta^c\right) B^{\mu\nu}_d - A^a_\mu \eta^b \eta^c A^{*\mu}_d$$
$$+ \left(-A^a_\mu A^b_\nu A^c_\rho + 6\eta^a B^{*b}_{\mu\nu} A^c_\rho + \eta^b \eta^c \eta^{*a}_{\mu\nu\rho}\right) \eta^{\mu\nu\rho}_d$$
$$- \frac{1}{3} \eta^a \eta^b \eta^c \eta^*_d + \left(-12A^a_\mu A^b_\nu B^{*c}_{\rho\lambda} + 12\eta^a B^{*b}_{\mu\nu} B^{*c}_{\rho\lambda}\right.$$
$$\left. - 8\eta^a \eta^{*b}_{\mu\nu\rho} A^c_\lambda + \eta^{*c}_{\mu\nu\rho\lambda} \eta^a \eta^b\right) \eta^{\mu\nu\rho\lambda}_d, \qquad \text{(A.6)}$$

$$U^{abc}_{d,e} = \left(H^{*\mu}_e A^{a\nu} \eta^b \eta^c + \frac{1}{6} C^{*\mu\nu}_e \eta^a \eta^b \eta^c\right) B_{d\mu\nu}$$
$$- \frac{1}{3} H^{*\mu}_e \eta^a \eta^b \eta^c A^*_{d\mu} + \left(-3H^{*\rho}_e \eta^a A^{b\mu} A^{c\nu}\right.$$
$$- 3H^{*\rho}_e \eta^a \eta^b B^{*c\mu\nu} + \frac{3}{2} C^{*\mu\nu}_e \eta^a \eta^b A^{c\rho}$$
$$\left. + \frac{1}{6} C^{*\mu\nu\rho}_e \eta^a \eta^b \eta^c\right) \eta_{d\mu\nu\rho} + \left(24A^{a\mu} H^{*\nu}_e \eta^b B^{*c\rho\lambda}\right.$$
$$+ 4H^{*\lambda}_e A^{a\mu} A^{b\nu} A^{c\rho} - 4H^{*\lambda}_e \eta^a \eta^b \eta^{*c\mu\nu\rho}$$
$$+ 6C^{*\mu\nu}_e \eta^a \eta^b B^{*c\rho\lambda} - 6C^{*\mu\nu}_e \eta^a A^{b\rho} A^{c\lambda}$$
$$\left. + 8C^{*\mu\nu\rho}_e \eta^a \eta^b A^{c\lambda} + \frac{1}{6} C^{*\mu\nu\rho\lambda}_e \eta^a \eta^b \eta^c\right) \eta_{d\mu\nu\rho\lambda}, \qquad \text{(A.7)}$$

$$U^{abc}_{d,ef} = \frac{1}{6} H^{*\mu}_e H^{*\nu}_f \eta^a \eta^b \eta^c B_{d\mu\nu} + \frac{3}{2} H^{*\mu}_e H^{*\nu}_f \eta^a \eta^b A^{c\rho} \eta_{d\mu\nu\rho}$$
$$+ \frac{1}{2} C^{*\mu\nu}_e H^{*\rho}_f \eta^a \eta^b \eta^c \eta_{d\mu\nu\rho} + \left(6H^{*\mu}_e H^{*\nu}_f \eta^a \eta^b B^{*c\rho\lambda}\right.$$
$$- 6H^{*\mu}_e H^{*\nu}_f \eta^a A^{b\rho} A^{c\lambda} + 6C^{*\mu\nu}_e H^{*\rho}_f \eta^a \eta^b A^{c\lambda} \qquad \text{(A.8)}$$
$$\left. + \frac{2}{3} C^{*\mu\nu\rho}_e H^{*\lambda}_f \eta^a \eta^b \eta^c + \frac{1}{2} C^{*\mu\nu}_e C^{*\rho\lambda}_f \eta^a \eta^b \eta^c\right) \eta_{d\mu\nu\rho\lambda},$$

$$U^{abc}_{d,efg} = \left(2H^{*\mu}_e H^{*\nu}_f H^{*\rho}_{\mathrm{g}} \eta^a \eta^b A^{c\lambda}\right.$$
$$\left. + C^{*\mu\nu}_e H^{*\rho}_f H^{*\lambda}_{\mathrm{g}} \eta^a \eta^b \eta^c\right) \eta_{d\mu\nu\rho\lambda}$$
$$+ \frac{1}{6} H^{*\mu}_e H^{*\nu}_f H^{*\rho}_{\mathrm{g}} \eta^a \eta^b \eta^c \eta_{d\mu\nu\rho}, \qquad \text{(A.9)}$$

$$U^{abc}_{d,efgh} = \frac{1}{6} H^{*\mu}_e H^{*\nu}_f H^{*\rho}_{\mathrm{g}} H^{*\lambda}_h \eta^a \eta^b \eta^c \eta_{d\mu\nu\rho\lambda}, \qquad \text{(A.10)}$$

$$K^{abcdf} = \frac{1}{8} \varepsilon^{\mu\nu\rho\lambda} \left[\left(\frac{1}{3!} A^a_\mu A^b_\nu - B^{*a}_{\mu\nu} \eta^b\right) A^c_\nu A^d_\rho \right. \qquad \text{(A.11)}$$
$$\left. + \frac{1}{3}\left(B^{*a}_{\mu\nu} B^{*b}_{\rho\lambda} - \frac{2}{3} \eta^{*a}_{\mu\nu\rho} A^b_\lambda + \frac{1}{4!} \eta^{*a}_{\mu\nu\rho\lambda} \eta^b\right) \eta^c \eta^d\right] \eta^f,$$

$$K^{abcdf}_e = \frac{1}{4!} \varepsilon^{\mu\nu\rho\lambda} \left[\frac{1}{2}\left(\frac{1}{5!} C^*_{e\mu\nu\rho\lambda} \eta^a + \frac{1}{3!} C^*_{e\mu\nu\rho} A^a_\lambda \right.\right. \qquad \text{(A.12)}$$
$$\left. + \frac{1}{2} C^*_{e\mu\nu} B^{*a}_{\rho\lambda} + \frac{1}{3} H^*_{e\mu} \eta^{*a}_{\nu\rho\lambda}\right) \eta^b \eta^c$$
$$\left. - H^*_{e\mu}\left(A^a_\nu A^b_\rho - 2B^{*a}_{\nu\rho} \eta^b\right) A^c_\lambda - \frac{1}{2} C^*_{e\mu\nu} A^a_\rho A^b_\lambda \eta^c\right] \eta^d \eta^f,$$

$$K^{abcdf}_{eg} = \frac{1}{2 \cdot 4!} \varepsilon^{\mu\nu\rho\lambda} \left[\frac{1}{2}\left(\frac{1}{15} H^*_{e\mu} C^*_{g\nu\rho\lambda} \eta^a \right.\right.$$
$$\left. + \frac{1}{20} C^*_{e\mu\nu} C^*_{g\rho\lambda} \eta^a + H^*_{e\mu} C^*_{g\nu\rho} A^a_\lambda\right) \eta^b$$
$$\left. - H^*_{e\mu} H^*_{g\nu}\left(A^a_\rho A^b_\lambda - 2B^{*a}_{\rho\lambda} \eta^b\right)\right] \eta^c \eta^d \eta^f, \qquad \text{(A.13)}$$

$$K^{abcdf}_{egh} = \frac{1}{4 \cdot 4!} \varepsilon^{\mu\nu\rho\lambda} H^*_{e\mu} H^*_{g\nu}$$
$$\times \left(\frac{1}{10} C^*_{h\rho\lambda} \eta^a + \frac{1}{3} H^*_{h\rho} A^a_\lambda\right) \eta^b \eta^c \eta^d \eta^f, \qquad \text{(A.14)}$$

$$K^{abcdf}_{eghl} = \frac{1}{2 \cdot 4! \cdot 5!} \varepsilon^{\mu\nu\rho\lambda} H^*_{e\mu} H^*_{g\nu} H^*_{h\rho} H^*_{l\lambda} \eta^a \eta^b \eta^c \eta^d \eta^f, \qquad \text{(A.15)}$$

$$K^a_b = 4\varepsilon^{\mu\nu\rho\lambda} \left[2\left(-C^a_{\mu\nu\rho\lambda} \eta^*_b + C^a_{\mu\nu\rho} A^*_{b\lambda}\right) + C^a_{\mu\nu} B_{b\rho\lambda}\right.$$
$$\left. - \left(\varphi^{*a} \eta_{b\mu\nu\rho\lambda} - H^a_\mu \eta_{b\nu\rho\lambda}\right)\right], \qquad \text{(A.16)}$$

$$K^a_{b,c} = 4\varepsilon^{\mu\nu\rho\lambda} \left[\eta_{b\mu\nu\rho\lambda}\left(C^a_{\sigma\tau\kappa\varsigma} C^{*\sigma\tau\kappa\varsigma}_c + C^a_{\sigma\tau\kappa} C^{*\sigma\tau\kappa}_c\right.\right.$$
$$\left. + C^a_{\sigma\tau} C^{*\sigma\tau}_c + H^a_\sigma H^{*\sigma}_c\right)$$
$$+ C^a_{\mu\nu\rho\lambda}\left(\eta_{b\sigma\tau\kappa} C^{*\sigma\tau\kappa}_c + B_{b\sigma\tau} C^{*\sigma\tau}_c - 2A^*_{b\sigma} H^{*\sigma}_c\right) \qquad \text{(A.17)}$$
$$\left. + \eta_{b\nu\rho\lambda}\left(3C^a_{\mu\sigma\tau} C^{*\sigma\tau}_c - 2C^a_{\mu\sigma} H^{*\sigma}_c\right) + 3B_{b\rho\lambda} C^a_{\mu\nu\sigma} H^{*\sigma}_c\right],$$

$$K^a_{b,cd} = 4\varepsilon^{\mu\nu\rho\lambda} \left[\eta_{b\mu\nu\rho\lambda}\left(C^a_{\sigma\tau\kappa\varsigma}\left(4H^{*\sigma}_c C^{*\tau\kappa\varsigma}_d + 3C^{*\sigma\tau}_c C^{*\kappa\varsigma}_d\right)\right.\right.$$
$$\left. + 3C^a_{\sigma\tau\kappa} H^{*\sigma}_c C^{*\tau\kappa}_d + C^a_{\sigma\tau} H^{*\sigma}_c H^{*\tau}_d\right)$$
$$\left. + C^a_{\mu\nu\rho\lambda}\left(3\eta_{b\sigma\tau\kappa} H^{*\sigma}_c C^{*\tau\kappa}_d + B_{b\sigma\tau} H^{*\sigma}_c H^{*\tau}_d\right)\right], \qquad \text{(A.18)}$$

$$K^a_{b,cde} = 4\varepsilon^{\mu\nu\rho\lambda} \left[\eta_{b\mu\nu\rho\lambda}\left(6C^a_{\sigma\tau\kappa\varsigma} H^{*\sigma}_c H^{*\tau}_d C^{*\kappa\varsigma}_e \right.\right. \qquad \text{(A.19)}$$
$$\left.\left. + C^a_{\sigma\tau\kappa} H^{*\sigma}_c H^{*\tau}_d H^{*\kappa}_e\right) + C^a_{\mu\nu\rho\lambda} \eta_{b\sigma\tau\kappa} H^{*\sigma}_c H^{*\tau}_d H^{*\kappa}_e\right],$$

$$K^a_{b,cdef} = 4\varepsilon_{\mu\nu\rho\lambda} \eta^{\mu\nu\rho\lambda}_b C^a_{\sigma\tau\kappa\varsigma} H^{*\sigma}_c H^{*\tau}_d H^{*\kappa}_e H^{*\varsigma}_f, \qquad \text{(A.20)}$$

$$K^c_{ab} = \varepsilon_{\mu\nu\rho\lambda} \left[-6\left(\eta^{\mu\nu\sigma}_a B^{\rho\lambda}_b A^c_\sigma + 3\eta^{\mu\sigma\tau}_a \eta^\nu_{b\sigma\tau} B^{*c\rho\lambda}\right)\right.$$
$$- 2\eta^{\mu\nu\rho\lambda}_a\left(\eta^{\sigma\tau\kappa\varsigma}_b \eta^{*c}_{\sigma\tau\kappa\varsigma} + 2\eta^{\sigma\tau\kappa}_b \eta^{*c}_{\sigma\tau\kappa} + 2B^{\sigma\tau}_b B^{*c}_{\sigma\tau} \right. \qquad \text{(A.21)}$$
$$\left.\left. - 2A^{*\sigma}_b A^c_\sigma - 2\eta^*_b \eta^c\right) + 4\eta^{\mu\nu\rho}_a A^{*\lambda}_b \eta^c - B^{\mu\nu}_a B^{\rho\lambda}_b \eta^c\right],$$

$$
\begin{aligned}
K^c_{ab,d} &= \varepsilon_{\mu\nu\rho\lambda}\Big[-9\eta_a^{\mu\sigma\tau}\eta^\nu_{b\sigma\tau}\Big(\eta^c C_d^{*\rho\lambda} - 2A^{c\rho}H_d^{*\lambda}\Big) \\
&- \eta_a^{\sigma\tau\kappa\varsigma}\eta_{b\sigma\tau\kappa\varsigma}\Big(\eta^c C_d^{*\mu\nu\rho\lambda} + 4C_d^{*\mu\nu\rho}A^{c\lambda} \\
&+ 12C_d^{*\mu\nu}B^{*c\rho\lambda} + 8H_d^{*\mu}\eta^{*c\nu\rho\lambda}\Big) + 6\eta_a^{\mu\nu\sigma}B_b^{\rho\lambda}\eta^c H^*_{d\sigma} \\
&- 2\eta_a^{\mu\nu\rho\lambda}\left(\eta_b^{\sigma\tau\kappa}\left(\eta^c C^*_{d\sigma\tau\kappa} + 3A^c_\kappa C^*_{d\sigma\tau} - 6B^{*c}_{\tau\kappa}H^*_{d\sigma}\right)\right. \\
&+ \left.\left. 2A_b^{*\sigma}\eta^c H^*_{d\sigma} + B_b^{\sigma\tau}\left(\eta^c C^*_{d\sigma\tau} + 2A^c_\tau H^*_{d\sigma}\right)\right)\right], \qquad \text{(A.22)}
\end{aligned}
$$

$$
\begin{aligned}
K^c_{ab,de} &= -\varepsilon_{\mu\nu\rho\lambda}\left[2\eta_a^{\mu\nu\rho\lambda}\right. \\
&\times \left(3\eta_b^{\sigma\tau\kappa}\left(H^*_{d\sigma}C^*_{e\tau\kappa}\eta^c + H^*_{d\sigma}H^*_{e\tau}A^c_\kappa\right) + B_b^{\sigma\tau}H^*_{d\sigma}H^*_{e\tau}\eta^c\right) \\
&+ \eta_a^{\sigma\tau\kappa\varsigma}\eta_{b\sigma\tau\kappa\varsigma}\left(\left(4H_d^{*\mu}C_e^{*\nu\rho\lambda} + 3C_d^{*\mu\nu}C_e^{*\rho\lambda}\right)\eta^c\right. \\
&+ \left. 12H_d^{*\mu}C_e^{*\nu\rho}A^{c\lambda} + 12H_d^{*\mu}H_e^{*\nu}B^{*c\rho\lambda}\right) \\
&+ \left. 9\eta_a^{\mu\sigma\tau}\eta^\nu_{b\sigma\tau}H_d^{*\rho}H_e^{*\lambda}\eta^c\right], \qquad \text{(A.23)}
\end{aligned}
$$

$$
\begin{aligned}
K^c_{ab,def} &= -2\varepsilon_{\mu\nu\rho\lambda}\left[\eta_a^{\sigma\tau\kappa\varsigma}\eta_{b\sigma\tau\kappa\varsigma}\right. \\
&\times \Big(3H_d^{*\mu}H_e^{*\nu}C_f^{*\rho\lambda}\eta^c + 2H_d^{*\mu}H_e^{*\nu}H_f^{*\rho}A^{c\lambda}\Big) \\
&+ \left.\eta_a^{\mu\nu\rho\lambda}\eta_b^{\sigma\tau\kappa}H^*_{d\sigma}H^*_{e\tau}H^*_{f\kappa}\eta^c\right], \qquad \text{(A.24)}
\end{aligned}
$$

$$
K^c_{ab,defg} = -\varepsilon_{\mu\nu\rho\lambda}\eta_a^{\sigma\tau\kappa\varsigma}\eta_{b\sigma\tau\kappa\varsigma}H_d^{*\mu}H_e^{*\nu}H_f^{*\rho}H_g^{*\lambda}\eta^c, \qquad \text{(A.25)}
$$

$$
\begin{aligned}
K^{abc}_d &= \eta_d^{\mu\nu\rho}\left(\eta^{*a}_{\mu\nu\rho}\eta^b\eta^c - 6B^{*a}_{\mu\nu}A^b_\rho\eta^c - A^a_\mu A^b_\nu A^c_\rho\right) \\
&- A_d^{*\mu}A^a_\mu\eta^b\eta^c + B_d^{\mu\nu}\left(B^{*a}_{\mu\nu}\eta^b\eta^c - A^a_\mu A^b_\nu\eta^c\right) \\
&+ \eta_d^{\mu\nu\rho\lambda}\left(\eta^{*a}_{\mu\nu\rho\lambda}\eta^b\eta^c - 4\eta^{*a}_{\mu\nu\rho}A^b_\lambda\eta^c\right. \qquad \text{(A.26)} \\
&+ \left. 12B^{*a}_{\mu\nu}\left(B^{*b}_{\rho\lambda}\eta^c - A^b_\rho A^c_\lambda\right)\right) - \frac{1}{6}\eta^*_d\eta^a\eta^b\eta^c,
\end{aligned}
$$

$$
\begin{aligned}
K^{abc}_{d,e} &= -\eta_d^{\mu\nu\rho}\left(\left(\left(\frac{1}{6}\eta^a C^*_{e\mu\nu\rho}\right.\right.\right. \\
&\qquad \left. -\frac{3}{2}A^a_\rho C^*_{e\mu\nu} + 3B^{*a}_{\nu\rho}H^*_{e\mu}\right)\eta^b\eta^c \\
&+ \left. 3A^a_\nu A^b_\rho\eta^c H^*_{e\mu}\right) + \eta_d^{\mu\nu\rho\lambda}\left(\left(\frac{1}{6}\eta^a C^*_{e\mu\nu\rho\lambda} + 2A^a_\lambda C^*_{e\mu\nu\rho}\right.\right. \\
&+ \left. 6B^{*a}_{\rho\lambda}C^*_{e\mu\nu} + 4\eta^{*a}_{\nu\rho\lambda}H^*_{e\mu}\right)\eta^b\eta^c - 2\left(3A^a_\rho A^b_\lambda\eta^c C^*_{e\mu\nu}\right. \\
&+ \left.\left. 12B^{*a}_{\nu\rho}A^b_\lambda\eta^c H^*_{e\mu} + 2A^a_\nu A^b_\rho A^c_\lambda H^*_{e\mu}\right)\right) \qquad \text{(A.27)} \\
&+ \left(-\frac{1}{3}A_d^{*\mu}\eta^a H^*_{e\mu} + B_d^{\mu\nu}\left(\frac{1}{6}\eta^a C^*_{e\mu\nu} + A^a_\nu H^*_{e\mu}\right)\right)\eta^b\eta^c,
\end{aligned}
$$

$$
\begin{aligned}
K^{abc}_{d,ef} &= -\frac{1}{2}\eta_d^{\mu\nu\rho}\left(\eta^a H^*_{e\mu}C^*_{f\nu\rho} - 3A^a_\rho H^*_{e\mu}C^*_{f\nu}\right)\eta^b\eta^c \\
&+ \eta_d^{\mu\nu\rho\lambda}\left(\left(\frac{2}{3}\eta^a H^*_{e\mu}C^*_{f\nu\rho\lambda} + \frac{1}{2}\eta^a C^*_{e\mu\nu}C^*_{f\rho\lambda}\right.\right. \\
&+ \left. 6A^a_\lambda H^*_{e\mu}C^*_{f\nu\rho} + \frac{1}{6}B^{*a}_{\rho\lambda}H^*_{e\mu}H^*_{f\nu}\right)\eta^b\eta^c \\
&- \left. 6A^a_\rho A^b_\lambda\eta^c H^*_{e\mu}H^*_{f\nu}\right) + \frac{1}{6}B_d^{\mu\nu}\eta^a\eta^b\eta^c H^*_{e\mu}H^*_{f\nu}, \qquad \text{(A.28)}
\end{aligned}
$$

$$
\begin{aligned}
K^{abc}_{d,efg} &= \left(\eta_d^{\mu\nu\rho\lambda}\left(\eta^a C^*_{g\rho\lambda} - 2A^a_\lambda H^*_{g\rho}\right) - \frac{1}{6}\eta_d^{\mu\nu\rho}\eta^a H^*_{g\rho}\right) \\
&\times \eta^b\eta^c H^*_{e\mu}H^*_{f\nu}, \qquad \text{(A.29)}
\end{aligned}
$$

$$
K^{abc}_{d,efgh} = \frac{1}{6}\eta_d^{\mu\nu\rho\lambda}\eta^a\eta^b\eta^c H^*_{e\mu}H^*_{f\nu}H^*_{g\rho}H^*_{h\lambda}. \qquad \text{(A.30)}
$$

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